Finite quasi-quantum groups of diagonal type

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Abstract. In this paper, we give a classification of finite-dimensional radically graded elementary quasi-Hopf algebras of diagonal type, or equivalently, finite-dimensional coradically graded pointed Majid algebras of diagonal type. By a Tannaka–Krein type duality, this determines a big class of pointed finite tensor categories. Some efficient methods of construction are also given.

1. Introduction

The classification problem of finite quasi-quantum groups is motivated mainly by the theory of finite tensor categories [15]. Among which, the classification of elementary quasi-Hopf algebras, or equivalently finite-dimensional pointed Majid algebras, has attracted much attention in the last one and a half decades. Quite a few examples and classification results of such algebras, and consequently the associated pointed finite tensor categories, were thus obtained, see e.g. [6, 11–13, 16]. In these studies, Etingof and Gelaki's novel idea of constructing genuine quasi-Hopf algebras from known pointed Hopf algebras plays a key role. This also builds a substantial connection from pointed finite tensor categories to the beautiful theory of finite-dimensional pointed Hopf algebras [1, 4], rather than just making the latter a role model in view of the obvious similarity.

The basic idea of Etingof and Gelaki in [11–13] is to embed a genuine elementary quasi-Hopf algebras into an elementary quasi-Hopf algebra, possibly up to twist equivalence. The crux of these constructions is that there is a resolution for any given 3-cocycle on a cyclic group, namely, for any 3-cocycle σ on $\mathbb{Z}_n = \langle g \mid g^n = 1 \rangle$, the pull-back $\pi^*(\sigma)$ along the natural projection $\pi : \mathbb{Z}_{n^2} \to \mathbb{Z}_n$ is a 3-coboundary on $\mathbb{Z}_{n^2}$. With this idea, the result of 3-cocycles on abelian groups of the form $\mathbb{Z}_m \times \mathbb{Z}_n$ obtained in [25] helps us to go a step forward in constructing new finite quasi-quantum groups. In our previous work [23], we gave a complete classification of finite-dimensional coradically graded pointed Majid algebras of rank 2. As a continuation of [23], the present paper aims to classify diagonal finite quasi-quantum groups of arbitrary rank.

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In this paper, we generalize the working spirit of [23] to the relatively general situation. Hence we need to solve four main problems as follows. The first problem, finding a resolution for any normalized 3-cocycle, lies basically in cohomology of finite abelian groups. By extending the idea of [25], we are able to give a unified and explicit formula for a complete set of representatives of normalized 3-cocycles on any finite abelian groups. Moreover, we show that a 3-cocycle is resolvable by a finite abelian group if and only if it is abelian and we give an explicit resolution if this is indeed the case. This is also the essential case where diagonal Nichols algebras occur for twisted Yetter–Drinfeld categories. For the second problem, to give a clear description of diagonal Nichols algebras in the twisted Yetter–Drinfeld category ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$, we transform them to those in the usual Yetter–Drinfeld category ${}^{G}_{G}\mathcal{YD}$ by a delicate manipulation, where G is a finite abelian group with canonical projection $\pi : G \to \mathbb{G}$ such that $\pi^*(\Phi)$ is a 3-coboundary on G. The possibility of such a transformation is guaranteed by the first step. Then by combining Heckenberger's classification of arithmetic root systems [20], we achieve a complete classification of diagonal Nichols algebras with arithmetic root systems in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. With the transformation, we can also reduce our third problem of generation into that of Nichols algebras in the usual Yetter–Drinfeld categories of finite abelian groups. With the help of Angiono's result [5], we extend the useful idea in [23] to the general situation and prove that finite-dimensional pointed Majid algebras of diagonal type are generated by group-likes and skew-primitive elements. The second and third steps together provide a complete classification of finite-dimensional graded pointed Majid algebras of diagonal type in a conceptual way. Finally, we shall need to turn the conceptual classification into an operable construction, our fourth problem. For any given finite abelian group with fixed 3-cocycle and a compatible arithmetic root system, the construction is essentially a computational problem of linear congruence equations. We find two efficient ways, for most cases, to generate series of new genuine finite-dimensional pointed Majid algebras.

Here is the layout of the paper. Section 2 is devoted to some preliminary materials. In Section 3, we provide an explicit formula for normalized 3-cocycles on finite abelian groups and give resolutions of the abelian ones via finite abelian groups. In Section 4, we give a complete classification of diagonal Nichols algebras with arithmetic root system in the twisted Yetter–Drinfeld category ${}^{G}_{G}\mathcal{YD}^{\Phi}$ with Φ nontrivial. Then, in Section 5, we classify in a conceptual way all the connected finite-dimensional graded pointed Majid algebras of diagonal type. Finally, in Section 6, we provide some methods to construct new genuine finite-dimensional pointed Majid algebras.

Throughout the paper, $\mathbf{k}$ is an algebraically closed field with characteristic zero and all linear spaces are over $\mathbf{k}$. A left (resp. bi-) G-comodule M, by definition, is a G-graded (resp. bigraded) space $M = \bigoplus_{g \in G} {}^g M$ (resp. $M = \bigoplus_{g,h \in G} {}^g M^h$). In general, we only deal with homogeneous elements unless stated otherwise. For convenience, if $X \in {}^g M$ (resp. $X \in {}^g M^h$) then we use its lowercase x to denote its degree, that is $x = g$ (resp. $x = gh^{-1}$). In accordance with our previous works [21–24], we only work on pointed Majid algebras. By taking linear dual, one has the version for elementary quasi-Hopf algebras.

2. Preliminaries

In this section, we recall some preliminary concepts, notations and facts. Clearly, there are some inevitable overlaps with the counterpart of [23]. For the completeness and for the convenience of the reader, we recall some materials presented already in [23].

2.1. Majid algebras. By definition, Majid algebras are exactly the dual of Drinfeld's quasi-Hopf algebras [9], and can be given as follows.

Definition 2.1. A Majid algebra is a coalgebra $(\mathbb{M}, \Delta, \varepsilon)$ equipped with a compatible quasi-algebra structure and a quasi-antipode. Namely, there exist two coalgebra homomorphisms

$$\mathbf{M} : \mathbb{M} \otimes \mathbb{M} \to \mathbb{M}, \quad a \otimes b \mapsto ab,$$
$$\mu : \mathbf{k} \to \mathbb{M}, \qquad \lambda \mapsto \lambda 1_{\mathbb{M}},$$

a convolution-invertible map $\Phi : \mathbb{M}^{\otimes 3} \to \mathbf{k}$ called associator, a coalgebra antimorphism $\mathcal{S} : \mathbb{M} \to \mathbb{M}$ and two functionals $\alpha, \beta : \mathbb{M} \to \mathbf{k}$ such that for all $a, b, c, d \in \mathbb{M}$ the following equalities hold:

$$a_1(b_1 c_1)\Phi(a_2, b_2, c_2) = \Phi(a_1, b_1, c_1)(a_2 b_2)c_2,$$
$$1_{\mathbb{M}} a = a = a 1_{\mathbb{M}},$$
$$\Phi(a_1, b_1, c_1 d_1)\Phi(a_2 b_2, c_2, d_2) = \Phi(b_1, c_1, d_1)\Phi(a_1, b_2 c_2, d_2)\Phi(a_2, b_3, c_3),$$
$$\Phi(a, 1_{\mathbb{M}}, b) = \varepsilon(a)\varepsilon(b).$$
$$\mathcal{S}(a_1)\alpha(a_2)a_3 = \alpha(a)1_{\mathbb{M}}, \quad a_1\beta(a_2)\mathcal{S}(a_3) = \beta(a)1_{\mathbb{M}},$$
$$\Phi(a_1, \mathcal{S}(a_3), a_5)\beta(a_2)\alpha(a_4) = \Phi^{-1}(\mathcal{S}(a_1), a_3, \mathcal{S}(a_5))\alpha(a_2)\beta(a_4) = \varepsilon(a).$$

Throughout we use the Sweedler sigma notation $\Delta(a) = a_1 \otimes a_2$ for the coproduct and $a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}$ for the result of the *n*-iterated application of Δ on a.

Example 2.2. Let G be a group and Φ a normalized 3-cocycle on G. It is well known that the group algebra $\mathbf{k}G$ is a Hopf algebra with $\Delta(g) = g \otimes g$, $\mathcal{S}(g) = g^{-1}$ and $\varepsilon(g) = 1$ for any $g \in G$. By extending Φ trilinearly, $\Phi : (\mathbf{k}G)^{\otimes 3} \to \mathbf{k}$ becomes a convolution-invertible map. Define two linear functions $\alpha, \beta : \mathbf{k}G \to \mathbf{k}$ just by

$$\alpha(g) := \varepsilon(g), \quad \beta(g) := \frac{1}{\Phi(g, g^{-1}, g)}$$

for any $g \in G$. Then kG together with these Φ, α and β becomes a Majid algebra. In the following, this resulting Majid algebra is denoted by $(\mathbf{k}G, \Phi)$.

Recall that a Majid algebra $\mathbb{M}$ is said to be *pointed* if the underlying coalgebra is so. Given a pointed Majid algebra $(\mathbb{M}, \Delta, \varepsilon, \mathbf{M}, \mu, \Phi, \mathcal{S}, \alpha, \beta)$, let $\{\mathbb{M}_n\}_{n \geq 0}$ be its coradical filtration, and

$$\mathrm{gr}\,\mathbb{M} = \mathbb{M}_0 \oplus \mathbb{M}_1/\mathbb{M}_0 \oplus \mathbb{M}_2/\mathbb{M}_1 \oplus \cdots$$

the corresponding coradically graded coalgebra. Then naturally $\operatorname{gr} \mathbb{M}$ inherits from $\mathbb{M}$ a graded Majid algebra structure. The corresponding graded associator $\operatorname{gr} \Phi$ satisfies $\operatorname{gr} \Phi(\bar{a}, \bar{b}, \bar{c}) = 0$ for all homogeneous $\bar{a}, \bar{b}, \bar{c} \in \operatorname{gr} \mathbb{M}$ unless they all lie in $\mathbb{M}_0$. A similar condition holds for $\operatorname{gr} \alpha$ and $\operatorname{gr} \beta$. In particular, $\mathbb{M}_0$ is a Majid subalgebra and it turns out to be the Majid algebra $(\mathbf{k}G, \operatorname{gr} \Phi)$ for $G = G(\mathbb{M})$, the set of group-like elements of $\mathbb{M}$. We call a pointed Majid algebra $\mathbb{M}$ *graded* if $\mathbb{M} \cong \operatorname{gr} \mathbb{M}$ as Majid algebras. We refer to [21] for more details on pointed Majid algebras.

Definition 2.3. Let $(\mathbb{M}, \Delta, \varepsilon, \mathrm{M}, \mu, \Phi, \mathcal{S}, \alpha, \beta)$ be a Majid algebra. A convolution-invertible linear map

$$J : \mathbb{M} \otimes \mathbb{M} \to \mathbf{k}$$

is called a twisting (or gauge transformation) on $\mathbb{M}$ if

$$J(h, 1) = \varepsilon(h) = J(1, h)$$

for all $h \in \mathbb{M}$.

Given a Majid algebra $\mathbb{M}$ and a twisting J, one can construct a new Majid algebra $\mathbb{M}^J$ as follows: $\mathbb{M}^J = \mathbb{M}$ as a coalgebra and the multiplication $\circ$ on $\mathbb{M}^J$ is given by

$$a \circ b := J(a_1, b_1) a_2 b_2 J^{-1}(a_3, b_3)$$

for all $a, b \in \mathbb{M}$. The associator Φ^J and the quasi-antipode $(\mathcal{S}^J, \alpha^J, \beta^J)$ are given as

$$\Phi^J(a, b, c) = J(b_1, c_1) J(a_1, b_2 c_2) \Phi(a_2, b_3, c_3) J^{-1}(a_3 b_4, c_4) J^{-1}(a_4, b_5),$$
$$\mathcal{S}^J = \mathcal{S}, \quad \alpha^J(a) = J^{-1}(\mathcal{S}(a_1), a_3) \alpha(a_2), \quad \beta^J(a) = J(a_1, \mathcal{S}(a_3)) \beta(a_2)$$

for all $a, b, c \in \mathbb{M}$.

Definition 2.4. Two Majid algebras $\mathbb{M}_1$ and $\mathbb{M}_2$ are called *twist equivalent* if there is a twisting J on $\mathbb{M}_1$ such that $\mathbb{M}_1^J \cong \mathbb{M}_2$ as Majid algebras. Denote $\mathbb{M}_1 \sim \mathbb{M}_2$ if $\mathbb{M}_1$ is twist equivalent to $\mathbb{M}_2$. We call a Majid algebra $\mathbb{M}$ *genuine* if it is not twist equivalent to a Hopf algebra.

2.2. Yetter–Drinfeld modules over $(\mathbf{k}G, \Phi)$. The definition of a Yetter–Drinfeld module over an arbitrary Majid algebra was already given in [7, Definition 3.1] and we recall it as follows.

Definition 2.5. Let $\mathbb{M}$ be a Majid algebra with associator Φ. A left-left Yetter–Drinfeld module over $\mathbb{M}$ is a triple $(V, \rho_V, \triangleright)$ such that

- (V, ρ_V) is a left comodule of $\mathbb{M}$ and we denote $\rho_V(v)$ by $v_{-1} \otimes v_0$ as usual;

- $\triangleright : M \otimes V \to V$ is a $\mathbf{k}$-linear map such that for all $h, l \in \mathbb{M}$ and $v \in V$,

$$(hl) \triangleright v = \frac{\Phi(h_2, (l_2 \triangleright v_0)_{-1}, l_3)}{\Phi(h_1, l_1, v_{-1}) \Phi((h_3 \triangleright (l_2 \triangleright v_0)_0)_{-1}, h_4, l_4)} (h_3 \triangleright (l_2 \triangleright v_0)_0)_0,$$
$$1_{\mathbb{M}} \triangleright v = v,$$
$$(h_1 \triangleright v)_{-1} h_2 \otimes (h_1 \triangleright v)_0 = h_1 v_{-1} \otimes (h_2 \triangleright v_0).$$

For our purpose, we want to describe a Yetter–Drinfeld module over Majid algebras of the form $(\mathbf{k}G, \Phi)$ with G a group. Assume that V is a left $\mathbf{k}G$-comodule with comodule structure map $\delta_L : V \to \mathbf{k}G \otimes V$. Define

$$^g V := \{v \in V \mid \delta_L(v) = g \otimes v\}$$

and thus

$$V = \bigoplus_{g \in G} {}^g V.$$

Here we call g the degree of the elements in $^g V$ and denote $\deg v = g$ for $v \in {}^g V$.

Definition 2.6. The left $\mathbf{k}G$-comodule (V, δ_L) is a left-left Yetter–Drinfeld module over the Majid algebra $\mathbb{M} = (\mathbf{k}G, \Phi)$ if there is a linear map $\rhd : G \otimes V \to V$ such that for all $e, f \in G$ and $v \in {}^g V$

$$e \rhd (f \rhd v) = \frac{\Phi(e, f, g)\Phi(efgf^{-1}e^{-1}, e, f)}{\Phi(e, fgf^{-1}, f)}(ef) \rhd v,$$

$$1_{\mathbb{M}} \rhd v = v,$$

(2.1) $\qquad e \rhd v \in {}^{ege^{-1}} V.$

The category of all left-left Yetter–Drinfeld modules over $(\mathbf{k}G, \Phi)$ is denoted by ${}^G_G \mathcal{YD}^\Phi$. Similarly, one can define left-right, right-left and right-right Yetter–Drinfeld modules over $(\mathbf{k}G, \Phi)$. As the familiar Hopf case, ${}^G_G \mathcal{YD}^\Phi$ is a braided tensor category. More precisely, for any $M, N \in {}^G_G \mathcal{YD}^\Phi$, the structure maps of $M \otimes N$ as a left-left Yetter–Drinfeld module are given by

$$\delta_L(m_g \otimes n_h) := gh \otimes m_g \otimes n_h,$$

$$x \rhd (m_g \otimes n_h) := \frac{\Phi(x, g, h)\Phi(xgx^{-1}, xhx^{-1}, x)}{\Phi(xgx^{-1}, x, h)} x \rhd m_g \otimes x \rhd n_h$$

for all $x, g, h \in G$ and $m_g \in {}^g M$, $n_h \in {}^h N$. The associativity constraint a and the braiding c of ${}^G_G \mathcal{YD}^\Phi$ are given respectively by

$$a((u_e \otimes v_f) \otimes w_g) = \Phi(e, f, g)^{-1} u_e \otimes (v_f \otimes w_g),$$

$$c(u_e \otimes v_f) = e \rhd v_f \otimes u_e$$

for all $e, f, g \in G$, $u_e \in {}^e U$, $v_f \in {}^f V$, $w_g \in {}^g W$ and $U, V, W \in {}^G_G \mathcal{YD}^\Phi$.

If moreover G is an abelian group, then we can simplify the above definition further. For this, define

$$\widetilde{\Phi}_g : G \times G \to \mathbf{k}^*, \quad (e, f) \mapsto \frac{\Phi(g, e, f)\Phi(e, f, g)}{\Phi(e, g, f)},$$

where $g \in G$. Direct computation shows that

$$\widetilde{\Phi}_g \in \mathbb{Z}^2(G, \mathbf{k}^*).$$

Definition 2.7. Assume that G is an abelian group. A left $\mathbf{k}G$-comodule V is a left-left Yetter–Drinfeld module over $(\mathbf{k}G, \Phi)$ if each $^g V$ is a projective G-representation with respect to the 2-cocycle $\widetilde{\Phi}_g$, namely the G-action $\triangleright$ on $^g V$ satisfies

$$e \triangleright (f \triangleright v) = \widetilde{\Phi}_g(e, f)(ef) \triangleright v \quad \text{for all } e, f \in G, \ v \in {}^g V.$$

Remark 2.8. For an arbitrary group G (may be not abelian), we say that a left-left Yetter–Drinfeld module V over $(\mathbf{k}G, \Phi)$ is of *diagonal* type if every $^g V$ is a projective G-representation and it is a direct sum of one-dimensional projective representations. In this case, the union of a nonzero element of each one-dimensional projective representation forms a basis of V, which is called a *canonical basis* of V in this paper. By equation (2.1), $^g V$ being a projective representation will imply that g lies in the center of G. Therefore, if V is diagonal then its *support group*, that is the subgroup generated by $\{g \mid {}^g V \neq 0\}$, must lie in the center of G and thus is an abelian group. We point out that not like the Hopf case, here even the whole G being abelian can *not* guarantee that every V is diagonal. It turns out that all $V \in {}^G_G \mathcal{YD}^\Phi$ are diagonal if and only if Φ is an abelian cocycle, see [28, 29], which is different from the Eilenberg–MacLane abelian cocycle [10]. We will discuss these cocycles in detail in Section 3.

2.3. Bosonization for pointed Majid algebras. The theory of bosonization in a broader context can be found in [27] in terms of braided diagrams. For our purpose, it is enough to focus on the situation of graded pointed Majid algebras. For the sake of completeness and later applications, we record in the following some explicit concepts, notations and results without proof.

In the rest of the paper, we always assume that

$$\mathbb{M} = \bigoplus_{i \in \mathbb{N}} \mathbb{M}_i$$

is a coradically graded connected pointed Majid algebra with unit 1. So $\mathbb{M}_0 = (\mathbf{k}G, \Phi)$ for some group G together with a 3-cocycle Φ on G. Let $\pi : \mathbb{M} \to \mathbb{M}_0$ be the canonical projection. Then $\mathbb{M}$ is a $\mathbf{k}G$-bicomodule naturally via

$$\delta_L := (\pi \otimes \mathrm{id})\Delta, \quad \delta_R := (\mathrm{id} \otimes \pi)\Delta.$$

Thus there is a G-bigrading on $\mathbb{M}$, that is,

$$\mathbb{M} = \bigoplus_{g,h \in G} {}^g \mathbb{M}^h,$$

where

$${}^g \mathbb{M}^h = \big\{ m \in \mathbb{M} \mid \delta_L(m) = g \otimes m, \ \delta_R(m) = m \otimes h \big\}.$$

As stated in the last paragraph of the introduction, we only deal with homogeneous elements with respect to this G-bigrading in this subsection. For example, whenever we write $\Delta(X) = X_1 \otimes X_2$, all X, X_1, X_2 are assumed homogeneous, and for any capital $X \in {}^g \mathbb{M}^h$, we use its lowercase x to denote gh^{-1}.

Define the coinvariant subalgebra of $\mathbb{M}$ by

$$\mathbb{R} := \big\{ m \in \mathbb{M} \mid (\mathrm{id} \otimes \pi)\Delta(m) = m \otimes 1 \big\}.$$

Clearly $1 \in \mathbb{R}$. There is a $(\mathbf{k}G, \Phi)$-action on $\mathbb{R}$ via

$$f \triangleright X := \frac{\Phi(fg, f^{-1}, f)}{\Phi(f, f^{-1}, f)}(f \cdot X) \cdot f^{-1}$$

for all $f, g \in G$ and $X \in {}^g\mathbb{R}$. Here $\cdot$ is the multiplication in $\mathbb{M}$. Then $(\mathbb{R}, \delta_L, \triangleright)$ is a left-left Yetter–Drinfeld module over $(\mathbf{k}G, \Phi)$.

Moreover, there are several natural operations on $\mathbb{R}$ inherited from $\mathbb{M}$ as follows:

$$\begin{aligned}
\mathrm{M} &: \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}, & (X, Y) &\mapsto XY := X \cdot Y, \\
u &: \mathbf{k} \to \mathbb{R}, & \lambda &\mapsto \lambda 1, \\
\Delta_{\mathbb{R}} &: \mathbb{R} \to \mathbb{R} \otimes \mathbb{R}, & X &\mapsto \Phi(x_1, x_2, x_2^{-1})X_1 \cdot x_2^{-1} \otimes X_2, \\
\varepsilon_{\mathbb{R}} &: \mathbb{R} \to \mathbf{k}, & \varepsilon_{\mathbb{R}} &:= \varepsilon|_{\mathbb{R}}, \\
\mathcal{S}_{\mathbb{R}} &: \mathbb{R} \to \mathbb{R}, & X &\mapsto \frac{1}{\Phi(x, x^{-1}, x)}x \cdot \mathcal{S}(X).
\end{aligned}$$

Then it is routine to verify that $(\mathbb{R}, \mathrm{M}, u, \Delta_{\mathbb{R}}, \varepsilon_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$ is a Hopf algebra in ${}^G_G\mathcal{YD}^\Phi$.

Conversely, let H be a Hopf algebra in ${}^G_G\mathcal{YD}^\Phi$. Since H is a left G-comodule, there is a G-grading on H:

$$H = \bigoplus_{x \in G} {}^xH,$$

where ${}^xH = \{X \in H \mid \delta_L(X) = x \otimes X\}$. As before, we only need to deal with G-homogeneous elements. As a convention, homogeneous elements in H are denoted by capital letters, say $X, Y, Z, \ldots$, and the associated degrees are denoted by their lower cases, say $x, y, z, \ldots$.

For our purpose, we also assume that H is $\mathbb{N}$-graded with $H_0 = \mathbf{k}$. If $X \in H_n$, then we say that X has length n. Moreover, we assume that both gradings are compatible in the sense that

$$H = \bigoplus_{g \in G} {}^gH = \bigoplus_{g \in G} \bigoplus_{n \in \mathbb{N}} {}^gH_n.$$

For example, the Hopf algebra $\mathbb{R}$ in ${}^G_G\mathcal{YD}^\Phi$ considered above satisfies these assumptions as $\mathbb{R} = \bigoplus_{i \in \mathbb{N}} \mathbb{R}_i$ is coradically graded. In this case, we call $\dim \mathbb{R}_1$ the rank of $\mathbb{R}$ and $\mathbb{M}$. For any $X \in H$, we write its comultiplication as

$$\Delta_H(X) = X_{(1)} \otimes X_{(2)}.$$

Lemma 2.9. *Keep the assumptions on H as above. Define on $H \otimes \mathbf{k}G$ a product by*

$$(X \otimes g)(Y \otimes h) = \frac{\Phi(xg, y, h)\Phi(x, y, g)}{\Phi(x, g, y)\Phi(xy, g, h)}X(g \triangleright Y) \otimes gh,$$

and a coproduct by

$$\Delta(X \otimes g) = \Phi(x_{(1)}, x_{(2)}, g)^{-1}(X_{(1)} \otimes x_{(2)}g) \otimes (X_{(2)} \otimes g).$$

Then $H \otimes \mathbf{k}G$ becomes a graded Majid algebra with a quasi-antipode $(\mathcal{S}, \alpha, \beta)$ given by

$$\mathcal{S}(X \otimes g) = \frac{\Phi(g^{-1}, g, g^{-1})}{\Phi(x^{-1}g^{-1}, xg, g^{-1})\Phi(x, g, g^{-1})}(1 \otimes x^{-1}g^{-1})(\mathcal{S}_H(X) \otimes 1),$$

$$\alpha(1 \otimes g) = 1, \quad \alpha(X \otimes g) = 0,$$

$$\beta(1 \otimes g) = \Phi(g, g^{-1}, g)^{-1}, \quad \beta(X \otimes g) = 0,$$

where $g, h \in G$ and X, Y are homogeneous elements of length ≥ 1.

In the following, by $H\#\mathbf{k}G$ we denote the resulting Majid algebra defined on $H \otimes \mathbf{k}G$.

Proposition 2.10. *Let* $\mathbb{M}$ *and* $\mathbb{R}$ *be as before, and let* $\mathbb{R}\#\mathbf{k}G$ *be the Majid algebra as defined in the previous proposition. Then the map*

$$F : \mathbb{R}\#\mathbf{k}G \to \mathbb{M}, \quad X \otimes g \mapsto Xg$$

is an isomorphism of Majid algebras.

2.4. Nichols algebras in $_G^G\mathcal{YD}^\Phi$. Nichols algebras can be defined by various equivalent ways, see for example [3]. Here we adopt the defining method in terms of the universal property. Roughly, Nichols algebras are the analogue of the usual symmetric algebras in more general braided tensor categories.

Let V be a nonzero object in $_G^G\mathcal{YD}^\Phi$. By $T_\Phi(V)$ we denote the tensor algebra in $_G^G\mathcal{YD}^\Phi$ generated freely by V. It is clear that $T_\Phi(V)$ is isomorphic to $\bigoplus_{n\geq 0} V^{\otimes \vec{n}}$ as a linear space, where

$$V^{\otimes \vec{n}} := \underbrace{(\cdots((}_{n-1} V \otimes V) \otimes V) \cdots \otimes V).$$

This induces a natural $\mathbb{N}$-graded structure on $T_\Phi(V)$. Define a comultiplication on $T_\Phi(V)$ by $\Delta(X) = X \otimes 1 + 1 \otimes X$ for all $X \in V$, a counit by $\varepsilon(X) = 0$, and an antipode by $S(X) = -X$. These provide a graded Hopf algebra structure on $T_\Phi(V)$ in the braided tensor category $_G^G\mathcal{YD}^\Phi$.

Definition 2.11. The Nichols algebra $\mathcal{B}(V)$ of V is defined to be the quotient Hopf algebra $T_\Phi(V)/I$ in $_G^G\mathcal{YD}^\Phi$, where I is the unique maximal graded Hopf ideal generated by homogeneous elements of degree greater than or equal to 2. Moreover, we call a Nichols algebra $\mathcal{B}(V)$ *diagonal* if V is a diagonal Yetter–Drinfeld module in $_G^G\mathcal{YD}^\Phi$.

To stress that our Nichols algebras may be non-associative in some occasions, we will call an associative Nichols algebra, e.g. $\mathcal{B}(V) \in _G^G\mathcal{YD}$, a usual Nichols algebra. The twisting process for Majid algebras can be transferred to Nichols algebras directly. In fact, let $(V, \rhd, \delta_L) \in _G^G\mathcal{YD}^\Phi$, and let J be a 2-cochain of G. Then we can define a new action $\rhd_J$ of G over V by

$$g \rhd_J X = \frac{J(g, x)}{J(x, g)} g \rhd X$$

for $X \in V$ and $g \in G$. We denote $(V, \rhd_J, \delta_L)$ by V^J and by definition we have

$$V^J \in _G^G\mathcal{YD}^{\Phi*\partial(J)}.$$

Moreover, there is a tensor equivalence $(F_J, \varphi_0, \varphi_2) : _G^G\mathcal{YD}^\Phi \to _G^G\mathcal{YD}^{\Phi*\partial(J)}$ which takes V to V^J and

$$\varphi_2(U, V) : (U \otimes V)^J \to U^J \otimes V^J, \quad Y \otimes Z \mapsto J(y, z)^{-1} Y \otimes Z$$

for $Y \in U, Z \in V$.

Let $\mathcal{B}(V)$ be a usual Nichols algebra in $_G^G\mathcal{YD}$. It is clear that $\mathcal{B}(V)^J$ is a Hopf algebra in $_G^G\mathcal{YD}^{\partial J}$ with multiplication $\circ$ determined by

$$X \circ Y = J(x, y)XY$$

for all homogeneous elements $X, Y \in \mathcal{B}(V)$, here $x = \deg X$, $y = \deg Y$. Using the same terminology as for Majid algebras or quasi-Hopf algebras, we say that $\mathcal{B}(V)$ and $\mathcal{B}(V)^J$ are twist equivalent. The following result is obvious, but important for our exposition.

Lemma 2.12. *The twisting $\mathcal{B}(V)^J$ of $\mathcal{B}(V)$ is a Nichols algebra in $^G_G \mathcal{YD}^{\partial J}$ and $\mathcal{B}(V)^J \cong \mathcal{B}(V^J)$.*

2.5. Arithmetic root systems and generalized Dynkin diagrams. Arithmetic root systems are invariants of Nichols algebras of diagonal type with certain finiteness property. A complete classification of arithmetic root systems was given by Heckenberger [20]. This is a crucial ingredient for the classification program of finite-dimensional pointed Hopf algebras, and turns out to be equally important in the broader situation of pointed Majid algebras.

Suppose $\mathcal{B}(V)$ is a usual Nichols algebra of diagonal type in $^G_G \mathcal{YD}$. Let $\{X_i \mid 1 \leq i \leq n\}$ be a canonical basis of V with $\delta_L(X_i) = h_i \otimes X_i$. The structure constants of $\mathcal{B}(V)$ are $\{q_{ij} \mid 1 \leq i, j \leq n\}$ such that $h_i \triangleright X_j = q_{ij} X_j$. Let $E = \{e_i \mid 1 \leq i \leq n\}$ be a canonical basis of $\mathbb{Z}^n$, and χ a bicharacter of $\mathbb{Z}^n$ determined by $\chi(e_i, e_j) = q_{ij}$. As defined in [17, Section 3], $\triangle^+(\mathcal{B}(V))$ is the set of degrees of the (restricted) Poincaré–Birkhoff–Witt generators counted with multiplicities, and

$$\triangle(\mathcal{B}(V)) := \triangle^+(\mathcal{B}(V)) \cup -\triangle^+(\mathcal{B}(V)),$$

which is called the root system of $\mathcal{B}(V)$. Moreover, the triple $(\triangle = \triangle(\mathcal{B}(V)), \chi, E)$ is called an arithmetic root system of $\mathcal{B}(V)$ if the corresponding Weyl groupoid $W_{\chi,E}$ is full and finite, see [19, Sections 2 and 3]. In this case, we denote this arithmetic root system by $\triangle(\mathcal{B}(V))_{\chi,E}$ for brevity. If there is another arithmetic root system $\triangle_{\chi',E'}$, and an isomorphism $\tau : \mathbb{Z}^n \to \mathbb{Z}^n$ such that

$$\tau(E) = E', \quad \chi'(\tau(e), \tau(e)) = \chi(e, e),$$
$$\chi'(\tau(e_1), \tau(e_2))\chi'(\tau(e_2), \tau(e_1)) = \chi(e_1, e_2)\chi(e_2, e_1),$$

then we say that $\triangle_{\chi,E}$ and $\triangle_{\chi',E'}$ are twist equivalent.

A generalized Dynkin diagram is an invariant of arithmetic root systems, and it can determine arithmetic root systems up to twist equivalence.

Definition 2.13. The generalized Dynkin diagram of an arithmetic root system $\triangle_{\chi,E}$ is a non-directed graph $\mathcal{D}_{\chi,E}$ with the following properties:

(1) There is a bijective map ϕ from $I = \{1, 2, \ldots, n\}$ to the set of vertices of $\mathcal{D}_{\chi,E}$.

(2) For all $1 \leq i \leq n$, the vertex $\phi(i)$ is labelled by q_{ii}.

(3) For all $1 \leq i, j \leq n$, the number n_{ij} of edges between $\phi(i)$ and $\phi(j)$ is either 0 or 1. If $i = j$ or $q_{ij}q_{ji} = 1$ then $n_{ij} = 0$, otherwise $n_{ij} = 1$ and the edge is labelled by $\widetilde{q_{ij}} = q_{ij}q_{ji}$ for $1 \leq i < j \leq n$.

An arithmetic root system is called connected provided the corresponding generalized Dynkin diagram $\mathcal{D}_{\chi,E}$ is connected. All the connected arithmetic root systems are classified and the corresponding generalized Dynkin diagrams are listed in [18, 20].

3. Normalized 3-cocycles over finite abelian groups

The aim of this section is threefold: Firstly, we will give a unified formula for 3-cocycles over a finite abelian group. Secondly, we want to develop a method to determine when a 3-cocycle is a coboundary. At last, we want to discuss the "resolution" problem, i.e., for every 3-cocycle Φ on $\mathbb{Z}_{\mathrm{m}_1} \times \cdots \times \mathbb{Z}_{\mathrm{m}_n}$, is there a bigger abelian group G together with a group epimorphism $\pi : G \to \mathbb{Z}_{\mathrm{m}_1} \times \cdots \times \mathbb{Z}_{\mathrm{m}_n}$ such that the pull-back $\pi^*(\Phi)$ is a coboundary on G? By this discussion, we find that there are essential differences between different 3-cocycles and we get the definition of abelian cocycles again, which was already discussed by Ng [29] and Mason–Ng [28].

3.1. A unified formula for 3-cocycles. Let G be a group and $(B_\bullet, \partial_\bullet)$ its bar resolution. By applying $\mathrm{Hom}_{\mathbb{Z}G}(-, \mathbf{k}^*)$, we get a complex $(B_\bullet^*, \partial_\bullet^*)$, where $\mathbf{k}^* = \mathbf{k} \setminus \{0\}$ is a trivial G-module.

Now let G be a finite abelian group. Thus $G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}$. For every $\mathbb{Z}_{m_i}$, we fix a generator g_i throughout this paper for $1 \leq i \leq k$. It is known, see e.g. [30, Section 6.2], that the following periodic sequence is a projective resolution for the trivial $\mathbb{Z}_{m_i}$-module $\mathbb{Z}$:

$$
(3.1) \qquad \cdots \longrightarrow \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{T_i} \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{N_i} \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{T_i} \mathbb{Z}\mathbb{Z}_{m_i} \xrightarrow{\varepsilon_i} \mathbb{Z} \to 0,
$$

where $T_i = g_i - 1$, $N_i = \sum_{j=0}^{m_i-1} g_i^j$ and ε_i is the augmentation map.

We want to form the tensor product of these periodic resolutions and get a resolution for the group $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}$. For the reader's convenience, we take the case $k = 2$ as an example to explain our construction at first and then provide the general form. In order to keep consistency with the notations used in [30, (2.7.1)], we rewrite the sequences (3.1) for $i = 1, 2$ in the following forms:

$$
\cdots \to P_3 \xrightarrow{T_1} P_2 \xrightarrow{N_1} P_1 \xrightarrow{T_1} P_0 \xrightarrow{\varepsilon_1} \mathbb{Z} \to 0,
$$

$$
\cdots \to Q_3 \xrightarrow{T_2} Q_2 \xrightarrow{N_2} Q_1 \xrightarrow{T_2} Q_0 \xrightarrow{\varepsilon_2} \mathbb{Z} \to 0.
$$

We use $P_\bullet \to \mathbb{Z} \to 0$ and $Q_\bullet \to \mathbb{Z} \to 0$ to denote them for short. Now we consider the tensor product over $\mathbb{Z}$ and for any nonnegative integers i, j we have

$$
P_i \otimes_{\mathbb{Z}} Q_j = \mathbb{Z}\mathbb{Z}_{m_1} \otimes_{\mathbb{Z}} \mathbb{Z}\mathbb{Z}_{m_2} \cong \mathbb{Z}(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) = \mathbb{Z}G
$$

which is a free $\mathbb{Z}G$-module of rank one. In order to remember the positions of $P_i \otimes_{\mathbb{Z}} Q_j$ in the double complex $D_{\bullet\bullet} := P_\bullet \otimes_{\mathbb{Z}} Q_\bullet$, we denote this free ZG-module by $(\mathbb{Z}G)\Psi(i, j)$, i.e.,

$$
(3.2) \qquad P_i \otimes_{\mathbb{Z}} Q_j = (\mathbb{Z}G)\Psi(i, j).
$$

Next we form the total complex $\mathrm{Tot}(D_{\bullet\bullet})$ of $P_\bullet \otimes_{\mathbb{Z}} Q_\bullet$ and want to give its differentials d clearly. By definition, we know that the horizontal differential of $P_i \otimes_{\mathbb{Z}} Q_j$ is just $d^1 \otimes 1$ where d^1 is the differential of $P_\bullet$ and the vertical differential is $(-1)^i 1 \otimes d^2$ where d^2 is the differential of $Q_\bullet$. Therefore,

$$
d(P_i \otimes_{\mathbb{Z}} Q_j) = d^1(P_i) \otimes_{\mathbb{Z}} Q_j + (-1)^i P_i \otimes_{\mathbb{Z}} d^2(Q_j).
$$

Using the convention of (3.2), we can describe the differential more explicitly. To this aim, define two morphisms d_1, d_2 of $\mathbb{Z}G$-modules through

$$d_1(\Psi(i, j)) = \begin{cases} 0 & \text{if } i = 0, \\ N_1 \Psi(i - 1, j) & \text{if } 0 \neq i \text{ even}, \\ T_1 \Psi(i - 1, j) & \text{if } i \text{ odd}, \end{cases}$$

$$d_2(\Psi(i, j)) = \begin{cases} 0 & \text{if } j = 0, \\ (-1)^i N_2 \Psi(i, j - 1) & \text{if } 0 \neq j \text{ even}, \\ (-1)^i T_2 \Psi(i, j - 1) & \text{if } j \text{ odd}. \end{cases}$$

It is not hard to see that $d = d_1 + d_2$. In one word, the total complex $\mathrm{Tot}(D_{\bullet\bullet})$ can be described as follows:

$$\mathrm{Tot}(D_{\bullet\bullet})_n = \bigoplus_{i+j=n} (\mathbb{Z}G)\Psi(i, j), \quad d = d_1 + d_2.$$

In general, let $K_\bullet$ be the following complex of projective (in fact, free) $\mathbb{Z}G$-modules. For each sequence $a_1, \ldots, a_k$ of nonnegative integers, let $\Psi(a_1, \ldots, a_k)$ be a free generator in degree $a_1 + \cdots + a_k$. Define

$$K_m := \bigoplus_{a_1+\cdots+a_k=m} (\mathbb{Z}G)\Psi(a_1, \ldots, a_k),$$

and

$$d_i(\Psi(a_1, \ldots, a_k)) = \begin{cases} 0 & \text{if } a_i = 0, \\ (-1)^{\sum_{l<i} a_l} N_i \Psi(a_1, \ldots, a_i - 1, \ldots, a_k) & \text{if } 0 \neq a_i \text{ even}, \\ (-1)^{\sum_{l<i} a_l} T_i \Psi(a_1, \ldots, a_i - 1, \ldots, a_k) & \text{if } a_i \text{ odd}, \end{cases}$$

for $1 \leq i \leq k$. The differential d is defined to be $d_1 + \cdots + d_k$. Now we can form a complex

$$(3.3) \qquad\qquad K_\bullet \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where ε denotes the augmentation map.

Lemma 3.1. *The complex* (3.3) *is a free resolution of the trivial* $\mathbb{Z}G$-module $\mathbb{Z}$.

Proof. Due to our construction, $(K_\bullet, d)$ is exactly the tensor product of the complexes (3.1). Therefore by the Künneth formula for complexes [30, (3.6.3)] we know that $K_\bullet$ is exact. Thus the only task is to show that $\mathrm{Ker}\, \varepsilon = \mathrm{Im}\, d\,|_{K_1}$, but this is clear. □

For convenience, we fix the following notations.

For any $1 \leq r \leq k$, define $\Psi_r := \Psi(0, \ldots, 1, \ldots, 0)$ where 1 lies in the r-th position. For any $1 \leq r \leq s \leq k$, define $\Psi_{r,s} := \Psi(0, \ldots, 1, \ldots, 1, \ldots, 0)$ where 1 lies in both the r-th and the s-th position if $r < s$ and $\Psi_{r,r} := \Psi(0, \ldots, 2, \ldots, 0)$ where 2 lies in the r-th position. Similarly, one can define $\Psi_{r,s,t}, \Psi_{r,s,s}, \Psi_{r,r,s}$ and $\Psi_{r,r,r}$ for $1 \leq r \leq k$, $1 \leq r < s \leq k$ and $1 \leq r < s < t \leq k$.

One could even define $\Psi_{i,j,s,t}$, $\Psi_{i,i,j,s}$, $\Psi_{i,j,s,s}$, $\Psi_{i,j,j,s}$, $\Psi_{i,i,j,j}$, $\Psi_{i,i,i,j}$, $\Psi_{i,j,j,j}$, and $\Psi_{i,i,i,i}$ for $1 \leq i \leq k$, $1 \leq i < j \leq k$, $1 \leq i < j < s \leq k$ and $1 \leq i < j < s < t \leq k$, respectively. Now it is clear that any cochain $f \in \mathrm{Hom}_{\mathbb{Z}G}(K_3, k^*)$ is uniquely determined by its values on $\Psi_{r,s,t}$, $\Psi_{r,s,s}$, $\Psi_{r,r,s}$ and $\Psi_{r,r,r}$ for $1 \leq r \leq k$, $1 \leq r < s \leq k$ and $1 \leq r < s < t \leq k$. For such numbers, we let

$$f_{r,s,t} = f(\Psi_{r,s,t}), \quad f_{r,s,s} = f(\Psi_{r,s,s}), \quad f_{r,r,s} = f(\Psi_{r,r,s}), \quad f_{r,r,r} = f(\Psi_{r,r,r}).$$

Lemma 3.2. *The 3-cochain $f \in \mathrm{Hom}_{\mathbb{Z}G}(K_3, \mathbf{k}^*)$ is a cocycle if and only if for all $1 \leq r \leq k$, $1 \leq r < s \leq k$ and $1 \leq r < s < t \leq k$,*

$$(3.4) \qquad f_{r,r,r}^{m_r} = 1, \quad f_{r,s,s}^{m_r} f_{r,r,s}^{m_s} = 1, \quad f_{r,s,t}^{m_r} = f_{r,s,t}^{m_s} = f_{r,s,t}^{m_t} = 1.$$

Proof. The proof follows by direct computations. By definition, the cochain f is a 3-cocycle if and only if $1 = d^*(f)(\Psi_{i,j,s,t}) = f(d(\Psi_{i,j,s,t}))$ for all $1 \leq i \leq j \leq s \leq t \leq k$. For any $a \in \mathbf{k}^*$, it is clear that $T_i \cdot a = 1$ since $\mathbf{k}^*$ is considered as a trivial G-module. Therefore we only need to consider the condition $1 = d^*(f)(\Psi_{i,j,s,t})$ in the cases: $i = j = s = t$, $i = j < s < t$, $i < j = s < t$, $i < j < s = t$ and $i = j < s = t$, respectively.

In case $i = j = s = t$, we have

$$1 = d^*(f)(\Psi_{i,i,i,i}) = f(N_i \Psi_{i,i,i}) = N_i \cdot f_{i,i,i} = f_{i,i,i}^{m_i}.$$

Similarly, we have the following:

$$\begin{aligned} f_{i,s,t}^{m_i} &= 1 \quad \text{if } i = j < s < t, \\ f_{i,j,t}^{-m_j} &= 1 \quad \text{if } i < j = s < t, \\ f_{i,j,s}^{m_s} &= 1 \quad \text{if } i < j < s = t, \\ f_{i,s,s}^{m_i} f_{i,i,s}^{m_s} &= 1 \quad \text{if } i = j < s = t. \end{aligned}$$

Now it is easy to see that these relations are the same as in equation (3.4). □

Lemma 3.3. *The 3-cochain $f \in \mathrm{Hom}_{\mathbb{Z}G}(K_3, \mathbf{k}^*)$ is a coboundary if and only if for all $1 \leq i < j \leq k$, there are $g_{i,j} \in \mathbf{k}^*$ such that*

$$(3.5) \qquad f_{i,i,j} = g_{i,j}^{m_i}, \quad f_{i,j,j} = g_{i,j}^{-m_j}, \quad \text{and} \quad f_{l,l,l} = 1, \quad f_{r,s,t} = 1,$$

for $1 \leq l \leq k$ and $1 \leq r < s < t \leq k$.

Proof. By definition, f is a coboundary if and only if $f = d^*(g)$ for some 2-cochain $g \in \mathrm{Hom}_{\mathbb{Z}G}(K_2, \mathbf{k}^*)$. For any $1 \leq i \leq j \leq k$, let $g_{i,j} := g(\Psi_{i,j})$. Since $T_l \cdot a = 1$ for any $a \in \mathbf{k}^*$, we have $d^*(g)(\Psi_{r,s,t}) = d^*(g)(\Psi_{l,l,l}) = 1$ for $1 \leq r < s < t \leq k$ and $1 \leq l \leq k$. Now for all $1 \leq i < j \leq k$,

$$\begin{aligned} f_{i,i,j} &= d^*(g)(\Psi_{i,i,j}) = g(N_i \Psi_{i,j} + T_j \Psi_{i,i}) = g_{i,j}^{m_i}, \\ f_{i,j,j} &= d^*(g)(\Psi_{i,j,j}) = g(T_i \Psi_{j,j} - N_j \Psi_{i,j}) = g_{i,j}^{-m_j}. \end{aligned}$$
□

For a set of natural numbers $s_1, \ldots, s_t$, by $(s_1, \ldots, s_t)$ we denote their greatest common divisor.

Proposition 3.4. *One has*

$$H^3(G, \mathbf{k}^*) \cong \prod_{i=1}^{n} \mathbb{Z}_{m_i} \times \prod_{1 \le i < j \le n} \mathbb{Z}_{(m_i, m_j)} \times \prod_{1 \le i < j < k \le n} \mathbb{Z}_{(m_i, m_j, m_k)}.$$

Proof. By Lemma 3.2 and Lemma 3.3, for a 3-cocycle f one can assume that $f_{l,l,l}$ is an m_l-th root of unity and $f_{i,j,k}$ is an (m_i, m_j, m_k)-th root of unity for all $1 \le l \le n$ and $1 \le i < j < k \le n$. By Lemma 3.3, one can take

$$g_{i,j} = f_{i,j,j}^{-1/m_j}$$

and thus can assume that $f_{i,j,j} = 1$ and $g_{i,j}^{m_j} = 1$ for all $1 \le i < j \le n$. By $f_{i,j,j}^{m_i} f_{i,i,j}^{m_j} = 1$, one has $f_{i,i,j}^{m_j} = 1$. Therefore, $H^3(G, \mathbf{k}^*)$ must be a quotient group of

$$\prod_{i=1}^{n} \mathbb{Z}_{m_i} \times \prod_{1 \le i < j \le n} \mathbb{Z}_{m_j} \times \prod_{1 \le i < j < k \le n} \mathbb{Z}_{(m_i, m_j, m_k)}.$$

Using the second relation in (3.5), one may even assume that $f_{i,i,j}^{m_i} = 1$. So the proposition is proved. $\qquad\square$

For any natural number m, once and for all we fix ζ_m to be a primitive m-th root of unity.

Corollary 3.5. *The set*

$$\big\{ f \in \mathrm{Hom}_{\mathbb{Z}G}(K_3, \mathbf{k}^*) \mid f_{l,l,l} = \zeta_{m_l}^{a_l}, \ f_{i,i,j} = \zeta_{m_j}^{a_{ij}}, \ f_{i,j,j} = 1, \ f_{r,s,t} = \zeta_{(m_r, m_s, m_t)}^{a_{rst}}$$

$$\text{for } 1 \le l \le n, \ 1 \le i < j \le n, \ 1 \le r < s < t \le n, \ and$$

$$0 \le a_l < m_l, \ 0 \le a_{ij} < (m_i, m_j), \ 0 \le a_{rst} < (m_r, m_s, m_t) \big\}$$

is a complete set of representatives of 3-cocycles of the complex $(K_\bullet^*, d_\bullet^*)$.

Next, we want to construct a chain map. We need some more notations to present the chain map. For any positive integers s and t, let $[\frac{s}{t}]$ denote the integer part of $\frac{s}{t}$ and let s_t' denote the remainder of division of s by t. When there is no risk of confusion, we drop the subscript and write simply s'. The following observation is useful in later arguments.

Lemma 3.6. *For any three natural numbers s, t, r, one has*

$$\left[\frac{s + t_r'}{r} \right] = \left[\frac{s + t}{r} \right] - \left[\frac{t}{r} \right].$$

Proof. We calculate

$$\left[\frac{s + t_r'}{r} \right] = \left[\frac{s + t - [\frac{t}{r}]r}{r} \right] = \left[\frac{s + t}{r} \right] - \left[\frac{t}{r} \right]. \qquad\square$$

Now we are ready to give a chain map, up to the third term for our purpose, from the normalized bar resolution $(B_\bullet, \partial_\bullet)$ to the tensor resolution $(K_\bullet, d_\bullet)$. Recall that B_m is the free

$\mathbb{Z}G$-module on the set of all symbols $[h_1, \ldots, h_m]$ with $h_i \in G$ and $m \geq 1$. In case $m = 0$, the symbol $[\,]$ denotes $1 \in \mathbb{Z}G$ and the map $\partial_0 = \epsilon : B_0 \to \mathbb{Z}$ sends $[\,]$ to 1.

We define the following three morphisms of $\mathbb{Z}G$-modules:

$$F_1 : B_1 \to K_1, \quad [g_1^{i_1} \cdots g_n^{i_n}] \mapsto \sum_{s=1}^{n} \sum_{\alpha_s=0}^{i_s-1} g_1^{i_1} \cdots g_{s-1}^{i_{s-1}} g_s^{\alpha_s} \Psi_s,$$

$$F_2 : B_2 \to K_2,$$

$$[g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}] \mapsto \sum_{s=1}^{n} g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} \left[\frac{i_s + j_s}{m_s}\right] \Psi_{s,s}$$

$$- \sum_{1 \leq s < t \leq n} \sum_{\alpha_s=0}^{j_s-1} \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s} g_t^{\beta_t} \Psi_{s,t},$$

$$F_3 : B_3 \to K_3,$$

$$[g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}]$$

$$\mapsto \sum_{r=1}^{n} \left[\frac{j_r + k_r}{m_r}\right] g_1^{j_1+k_1} \cdots g_{r-1}^{j_{r-1}+k_{r-1}} \sum_{\beta_r=0}^{i_r-1} g_1^{i_1} \cdots g_{r-1}^{i_{r-1}} g_r^{\beta_r} \Psi_{r,r,r}$$

$$+ \sum_{1 \leq r < t \leq n} \left[\frac{j_r + k_r}{m_r}\right] g_1^{j_1+k_1} \cdots g_{r-1}^{j_{r-1}+k_{r-1}} \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_t^{\beta_t} \Psi_{r,r,t}$$

$$+ \sum_{1 \leq r < t \leq n} \left[\frac{i_t + j_t}{m_t}\right] g_1^{i_1+j_1} \cdots g_{t-1}^{i_{t-1}+j_{t-1}} \sum_{\gamma_r=0}^{k_r-1} g_1^{k_1} \cdots g_{r-1}^{k_{r-1}} g_r^{\gamma_r} \Psi_{r,t,t}$$

$$- \sum_{1 \leq r < s < t \leq n} \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_t^{\beta_t} \sum_{\alpha_s=0}^{j_s-1} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s} \sum_{\gamma_r=0}^{k_r-1} g_1^{k_1} \cdots g_{r-1}^{k_{r-1}} g_r^{\gamma_r} \Psi_{r,s,t}$$

for $0 \leq i_l, j_l, k_l < m_l$ and $1 \leq l \leq n$.

Proposition 3.7. *The following diagram is commutative:*

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & B_3 & \xrightarrow{\partial_3} & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\partial_1} & B_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
& & \downarrow{F_3} & & \downarrow{F_2} & & \downarrow{F_1} & & \| & & \| \\
\cdots & \longrightarrow & K_3 & \xrightarrow{d} & K_2 & \xrightarrow{d} & K_1 & \xrightarrow{d} & K_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0.
\end{array}
$$

Proof. The proof is by direct but very complicated computation. The essence of the proposition lies in figuring out the morphisms F_1, F_2 and F_3 in the first place. We hope that the proof may shed some light on the construction of them. The proof is naturally divided into three parts.

Claim 1: $dF_1 = \partial_1$. Take any generator $[g_1^{i_1} \cdots g_n^{i_n}] \in B_1$. Then

$$\partial_1([g_1^{i_1} \cdots g_n^{i_n}]) = (g_1^{i_1} \cdots g_n^{i_n} - 1)\Psi(0, \ldots, 0)$$

and

$$dF_1([g_1^{i_1} \cdots g_n^{i_n}]) = d\left(\sum_{s=1}^{n} \sum_{\alpha_s=0}^{i_s-1} g_1^{i_1} \cdots g_{s-1}^{i_{s-1}} g_s^{\alpha_s} \Psi_s \right)$$

$$= \sum_{s=1}^{n} \sum_{\alpha_s=0}^{i_s-1} g_1^{i_1} \cdots g_{s-1}^{i_{s-1}} g_s^{\alpha_s} (g_s - 1) \Psi(0, \dots, 0)$$

$$= \sum_{s=1}^{n} g_1^{i_1} \cdots g_{s-1}^{i_{s-1}} (g_s^{i_s} - 1) \Psi(0, \dots, 0)$$

$$= (g_1^{i_1} \cdots g_n^{i_n} - 1) \Psi(0, \dots, 0).$$

Claim 2: $dF_2 = F_1 \partial_2$.   For any generator $[g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}]$, we have

$$F_1 \partial_2([g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}])$$

$$= F_1\left(g_1^{i_1} \cdots g_n^{i_n} [g_1^{j_1} \cdots g_n^{j_n}] - [g_1^{i_1+j_1} \cdots g_n^{i_n+j_n}] + [g_1^{i_1} \cdots g_n^{i_n}] \right)$$

$$= g_1^{i_1} \cdots g_n^{i_n} \sum_{s=1}^{n} \sum_{\alpha_s=0}^{j_s-1} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s} \Psi_s$$

$$- \sum_{s=1}^{n} \sum_{\alpha_s=0}^{(i_s+j_s)'-1} g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} g_s^{\alpha_s} \Psi_s$$

$$+ \sum_{s=1}^{n} \sum_{\alpha_s=0}^{i_s-1} g_1^{i_1} \cdots g_{s-1}^{i_{s-1}} g_s^{\alpha_s} \Psi_s.$$

Fix any s, the coefficient of Ψ_s is

$$(3.6) \qquad g_1^{i_1} \cdots g_n^{i_n} \sum_{\alpha_s=0}^{j_s-1} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s}$$

$$- g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} \left(\sum_{\alpha_s=0}^{i_s+j_s-1} g_s^{\alpha_s} - \left[\frac{i_s + j_s}{m_s} \right] N_s \right)$$

$$+ \sum_{\alpha_s=0}^{i_s-1} g_1^{i_1} \cdots g_{s-1}^{i_{s-1}} g_s^{\alpha_s}.$$

Now consider dF_2. We have

$$dF_2([g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}])$$

$$= d\left(\sum_{s=1}^{n} g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} \left[\frac{i_s + j_s}{m_s} \right] \Psi_{s,s} \right)$$

$$- d\left(\sum_{1 \leq s < t \leq n} \sum_{\alpha_s=0}^{j_s-1} \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s} g_t^{\beta_t} \Psi_{s,t} \right).$$

In this expression, the coefficient of Ψ_s is

$$
g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} \left[\frac{i_s + j_s}{m_s} \right] N_s
$$

$$
- \sum_{1 \le t < s} \sum_{\beta_s=0}^{i_s-1} g_1^{i_1} \cdots g_{s-1}^{i_{s-1}} g_1^{j_1} \cdots g_{t-1}^{j_{t-1}} (g_t^{j_t} - 1) g_s^{\beta_s}
$$

$$
+ \sum_{s<t\le n} \sum_{\alpha_s=0}^{j_s-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s} (g_t^{i_t} - 1)
$$

$$
= g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} \left[\frac{i_s + j_s}{m_s} \right] N_s
$$

$$
- \sum_{\beta_s=0}^{i_s-1} g_1^{i_1} \cdots g_{s-1}^{i_{s-1}} (g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} - 1) g_s^{\beta_s}
$$

$$
+ \sum_{\alpha_s=0}^{j_s-1} (g_1^{i_1} \cdots g_n^{i_n} - g_1^{i_1} \cdots g_s^{i_s}) g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s}
$$

$$
= g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} \left[\frac{i_s + j_s}{m_s} \right] N_s
$$

$$
- \sum_{\beta_s=0}^{i_s+j_s-1} g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} g_s^{\beta_s}
$$

$$
+ \sum_{\alpha_s=0}^{j_s-1} g_1^{i_1} \cdots g_n^{i_n} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s} + \sum_{\beta_s=0}^{i_s-1} g_1^{i_1} \cdots g_{s-1}^{i_{s-1}} g_s^{\beta_s},
$$

which is clearly identical with (3.6). So we have $dF_2 = F_1 \partial_2$.

Claim 3: $dF_3 = F_2 \partial_3$. Similarly, for any generator $[g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}]$, we have

$$
F_2 \partial_3 ([g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}])
$$

$$
= F_2 \big(g_1^{i_1} \cdots g_n^{i_n} [g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}] - [g_1^{i_1+j_1} \cdots g_n^{i_n+j_n}, g_1^{k_1} \cdots g_n^{k_n}] \big)
$$

$$
+ F_2 \big([g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1+k_1} \cdots g_n^{j_n+k_n}] - [g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}] \big)
$$

$$
= g_1^{i_1} \cdots g_n^{i_n} \sum_{s=1}^{n} g_1^{j_1+k_1} \cdots g_{s-1}^{j_{s-1}+k_{s-1}} \left[\frac{j_s + k_s}{m_s} \right] \Psi_{s,s}
$$

$$
- g_1^{i_1} \cdots g_n^{i_n} \sum_{1\le s<t\le n} \sum_{\alpha_s=0}^{k_s-1} \sum_{\beta_t=0}^{j_t-1} g_1^{j_1} \cdots g_{t-1}^{j_{t-1}} g_1^{k_1} \cdots g_{s-1}^{k_{s-1}} g_s^{\alpha_s} g_t^{\beta_t} \Psi_{s,t}
$$

$$
- \sum_{s=1}^{n} g_1^{i_1+j_1+k_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}+k_{s-1}} \left[\frac{(i_s + j_s)' + k_s}{m_s} \right] \Psi_{s,s}
$$

$$
+ \sum_{1\le s<t\le n} \sum_{\alpha_s=0}^{k_s-1} \sum_{\beta_t=0}^{(i_t+j_t)'-1} g_1^{i_1+j_1} \cdots g_{t-1}^{i_{t-1}+j_{t-1}} g_1^{k_1} \cdots g_{s-1}^{k_{s-1}} g_s^{\alpha_s} g_t^{\beta_t} \Psi_{s,t}
$$

$$+ \sum_{s=1}^{n} g_1^{i_1+j_1+k_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}+k_{s-1}} \left[\frac{i_s + (j_s + k_s)'}{m_s} \right] \Psi_{s,s}$$

$$- \sum_{1 \le s < t \le n} \sum_{\alpha_s=0}^{(j_s+k_s)'-1} \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_1^{j_1+k_1} \cdots g_{s-1}^{j_{s-1}+k_{s-1}} g_s^{\alpha_s} g_t^{\beta_t} \Psi_{s,t}$$

$$- \sum_{s=1}^{n} g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} \left[\frac{i_s + j_s}{m_s} \right] \Psi_{s,s}$$

$$+ \sum_{1 \le s < t \le n} \sum_{\alpha_s=0}^{j_s-1} \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s} g_t^{\beta_t} \Psi_{s,t}.$$

Note that in $(i_s + j_s)'$ we drop the subscript m_s. In the previous expression, for any $1 \le s \le n$, the coefficient of $\Psi_{s,s}$ is

$$(3.7) \qquad g_1^{i_1} \cdots g_n^{i_n} g_1^{j_1+k_1} \cdots g_{s-1}^{j_{s-1}+k_{s-1}} \left[\frac{j_s + k_s}{m_s} \right]$$

$$+ g_1^{i_1+j_1+k_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}+k_{s-1}} \left(\left[\frac{i_s + j_s}{m_s} \right] - \left[\frac{j_s + k_s}{m_s} \right] \right)$$

$$- g_1^{i_1+j_1} \cdots g_{s-1}^{i_{s-1}+j_{s-1}} \left[\frac{i_s + j_s}{m_s} \right],$$

where Lemma 3.6 is applied. For any $1 \le s < t \le n$, the coefficient of $\Psi_{s,t}$ is

$$(3.8) \quad - g_1^{i_1} \cdots g_n^{i_n} \sum_{\alpha_s=0}^{k_s-1} \sum_{\beta_t=0}^{j_t-1} g_1^{j_1} \cdots g_{t-1}^{j_{t-1}} g_1^{k_1} \cdots g_{s-1}^{k_{s-1}} g_s^{\alpha_s} g_t^{\beta_t}$$

$$+ \sum_{\alpha_s=0}^{k_s-1} g_1^{i_1+j_1} \cdots g_{t-1}^{i_{t-1}+j_{t-1}} g_1^{k_1} \cdots g_{s-1}^{k_{s-1}} g_s^{\alpha_s} \left(\sum_{\beta_t=0}^{i_t+j_t-1} g_t^{\beta_t} - \left[\frac{i_t + j_t}{m_t} \right] N_t \right)$$

$$- \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_t^{\beta_t} g_1^{j_1+k_1} \cdots g_{s-1}^{j_{s-1}+k_{s-1}} \left(\sum_{\alpha_s=0}^{j_s+k_s-1} g_s^{\alpha_s} - \left[\frac{j_s + k_s}{m_s} \right] N_s \right)$$

$$+ \sum_{\alpha_s=0}^{j_s-1} \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s} g_t^{\beta_t}.$$

For dF_3, we have

$$dF_3([g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}])$$

$$= d \left(\sum_{r=1}^{n} \left[\frac{j_r + k_r}{m_r} \right] g_1^{j_1+k_1} \cdots g_{r-1}^{j_{r-1}+k_{r-1}} \sum_{\beta_r=0}^{i_r-1} g_1^{i_1} \cdots g_{r-1}^{i_{r-1}} g_r^{\beta_r} \Psi_{r,r,r} \right)$$

$$+ d \left(\sum_{1 \le r < t \le n} \left[\frac{j_r + k_r}{m_r} \right] g_1^{j_1+k_1} \cdots g_{r-1}^{j_{r-1}+k_{r-1}} \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_t^{\beta_t} \Psi_{r,r,t} \right)$$

$$+ d \left(\sum_{1 \le r < t \le n} \left[\frac{i_t + j_t}{m_t} \right] g_1^{i_1+j_1} \cdots g_{t-1}^{i_{t-1}+j_{t-1}} \sum_{\gamma_r=0}^{k_r-1} g_1^{k_1} \cdots g_{r-1}^{k_{r-1}} g_r^{\gamma_r} \Psi_{r,t,t} \right)$$

$$-d\Bigg(\sum_{1\le r<s<t\le n}\sum_{\beta_t=0}^{i_t-1}g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}g_t^{\beta_t}\sum_{\alpha_s=0}^{j_s-1}g_1^{j_1}\cdots g_{s-1}^{j_{s-1}}g_s^{\alpha_s}$$

$$\times\sum_{\gamma_r=0}^{k_r-1}g_1^{k_1}\cdots g_{r-1}^{k_{r-1}}g_r^{\gamma_r}\Psi_{r,s,t}\Bigg).$$

Note that the coefficient of $\Psi_{s,s}$ is

$$\left[\frac{j_s+k_s}{m_s}\right]g_1^{j_1+k_1}\cdots g_{s-1}^{j_{s-1}+k_{s-1}}g_1^{i_1}\cdots g_{s-1}^{i_{s-1}}(g_s^{i_s}-1)$$

$$+\sum_{s<t\le n}\left[\frac{j_s+k_s}{m_s}\right]g_1^{j_1+k_1}\cdots g_{r-1}^{j_{s-1}+k_{s-1}}g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}(g_t^{i_t}-1)$$

$$+\sum_{1\le r<s}\left[\frac{i_s+j_s}{m_s}\right]g_1^{i_1+j_1}\cdots g_{s-1}^{i_{s-1}+j_{s-1}}g_1^{k_1}\cdots g_{r-1}^{k_{r-1}}(g_r^{k_r}-1)$$

$$=\left[\frac{j_s+k_s}{m_s}\right]g_1^{j_1+k_1}\cdots g_{s-1}^{j_{s-1}+k_{s-1}}g_1^{i_1}\cdots g_{s-1}^{i_{s-1}}(g_s^{i_s}-1)$$

$$+\left[\frac{j_s+k_s}{m_s}\right]g_1^{j_1+k_1}\cdots g_{r-1}^{j_{s-1}+k_{s-1}}(g_1^{i_1}\cdots g_n^{i_n}-g_1^{i_1}\cdots g_s^{i_s})$$

$$+\left[\frac{i_s+j_s}{m_s}\right]g_1^{i_1+j_1}\cdots g_{s-1}^{i_{s-1}+j_{s-1}}(g_1^{k_1}\cdots g_{s-1}^{k_{s-1}}-1),$$

which clearly is equal to (3.7).

Finally, we consider the coefficient of $\Psi_{s,t}$ for $1\le s<t\le n$, which is

$$\left[\frac{j_s+k_s}{m_s}\right]N_s g_1^{j_1+k_1}\cdots g_{s-1}^{j_{s-1}+k_{s-1}}\sum_{\beta_t=0}^{i_t-1}g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}g_t^{\beta_t}$$

$$+\left[\frac{i_t+j_t}{m_t}\right]N_t g_1^{i_1+j_1}\cdots g_{t-1}^{i_{t-1}+j_{t-1}}\sum_{\gamma_s=0}^{k_s-1}g_1^{k_1}\cdots g_{s-1}^{k_{s-1}}g_s^{\gamma_s}$$

$$-\sum_{1\le r<s<t}\sum_{\beta_t=0}^{i_t-1}g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}g_t^{\beta_t}\sum_{\alpha_s=0}^{j_s-1}g_1^{j_1}\cdots g_{s-1}^{j_{s-1}}g_s^{\alpha_s}g_1^{k_1}\cdots g_{r-1}^{k_{r-1}}(g_r^{k_r}-1)$$

$$+\sum_{s<r<t}\sum_{\beta_t=0}^{i_t-1}g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}g_t^{\beta_t}g_1^{j_1}\cdots g_{r-1}^{j_{r-1}}(g_r^{j_r}-1)\sum_{\gamma_s=0}^{k_s-1}g_1^{k_1}\cdots g_{s-1}^{k_{s-1}}g_s^{\gamma_s}$$

$$-\sum_{s<t<r}g_1^{i_1}\cdots g_{r-1}^{i_{r-1}}(g_r^{i_r}-1)\sum_{\alpha_t=0}^{j_t-1}g_1^{j_1}\cdots g_{t-1}^{j_{t-1}}g_t^{\alpha_t}\sum_{\gamma_s=0}^{k_s-1}g_1^{k_1}\cdots g_{s-1}^{k_{s-1}}g_s^{\gamma_s}$$

$$=\left[\frac{j_s+k_s}{m_s}\right]N_s g_1^{j_1+k_1}\cdots g_{s-1}^{j_{s-1}+k_{s-1}}\sum_{\beta_t=0}^{i_t-1}g_1^{i_1}\cdots g_{t-1}^{i_{t-1}}g_t^{\beta_t}$$

$$+\left[\frac{i_t+j_t}{m_t}\right]N_t g_1^{i_1+j_1}\cdots g_{t-1}^{i_{t-1}+j_{t-1}}\sum_{\gamma_s=0}^{k_s-1}g_1^{k_1}\cdots g_{s-1}^{k_{s-1}}g_s^{\gamma_s}$$

$$- \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_t^{\beta_t} \sum_{\alpha_s=0}^{j_s-1} g_1^{j_1} \cdots g_{s-1}^{j_{s-1}} g_s^{\alpha_s} (g_1^{k_1} \cdots g_{s-1}^{k_{s-1}} - 1)$$

$$+ \sum_{\beta_t=0}^{i_t-1} g_1^{i_1} \cdots g_{t-1}^{i_{t-1}} g_t^{\beta_t} (g_1^{j_1} \cdots g_{t-1}^{j_{t-1}} - g_1^{j_1} \cdots g_s^{j_s}) \sum_{\gamma_s=0}^{k_s-1} g_1^{k_1} \cdots g_{s-1}^{k_{s-1}} g_s^{\gamma_s}$$

$$- (g_1^{i_1} \cdots g_n^{i_n} - g_1^{i_1} \cdots g_t^{i_t}) \sum_{\alpha_t=0}^{j_t-1} g_1^{j_1} \cdots g_{t-1}^{j_{t-1}} g_t^{\alpha_t} \sum_{\gamma_s=0}^{k_s-1} g_1^{k_1} \cdots g_{s-1}^{k_{s-1}} g_s^{\gamma_s}.$$

It is not hard to see that this is equal to (3.8). Therefore, $dF_3 = F_2 \partial_3$.

The proof is completed. $\square$

Now we are able to accomplish the main task with a help of the results obtained above. Define A to be the set of all sequences like

$$(3.9) \quad (a_1, \ldots, a_l, \ldots, a_n, a_{12}, \ldots, a_{ij}, \ldots, a_{n-1,n}, a_{123}, \ldots, a_{rst}, \ldots, a_{n-2,n-1,n})$$

such that $0 \leq a_l < m_l$, $0 \leq a_{ij} < (m_i, m_j)$ and $0 \leq a_{rst} < (m_r, m_s, m_t)$ for $1 \leq l \leq n$, $1 \leq i < j \leq n$, $1 \leq r < s < t \leq n$ where a_{ij} and a_{rst} are ordered by the lexicographic order. In the following, the sequence (3.9) is denoted by $\underline{a}$ for short.

For any $\underline{a} \in A$, define a $\mathbb{Z}G$-module morphism:

$$(3.10) \quad \Phi_{\underline{a}} : B_3 \to \mathbf{k}^*,$$
$$[g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}]$$
$$\mapsto \prod_{l=1}^n \zeta_{m_l}^{a_l i_l [\frac{j_l+k_l}{m_l}]} \prod_{1 \leq s < t \leq n} \zeta_{m_t}^{a_{st} i_t [\frac{j_s+k_s}{m_s}]} \prod_{1 \leq r < s < t \leq n} \zeta_{(m_r, m_s, m_t)}^{-a_{rst} k_r j_s i_t}.$$

Proposition 3.8. *Suppose that* $\mathbf{k}$ *is an algebraically closed field of characteristic zero and* $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$. *Then* $\{\Phi_{\underline{a}} \mid \underline{a} \in A\}$ *is a complete set of representatives of normalized* 3-*cocycles on* G.

Proof. This is a direct consequence of Corollary 3.5 and the definition of the map F_3 given in Proposition 3.7. $\square$

3.2. 3-coboundary. Later on, we will encounter the following problem: Given a 3-cocycle of the complex $(B_\bullet^*, \partial_\bullet^*)$, we have to determine whether it is a 3-coboundary or not. In this subsection, we want to solve this problem in case G is a finite abelian group. In fact, Lemma 3.3 already provides us an easy way. For the bar resolution, it is sufficient to give a chain map from $(K_\bullet, d_\bullet)$ to $(B_\bullet, \partial_\bullet)$, which is a kind of inverse of the chain map defined in the previous subsection and thus becomes much simpler. We use the following three morphisms of $\mathbb{Z}G$-modules defined in [26, Section 2]:

$$F_1 : K_1 \to B_1, \quad \Psi_r \mapsto [g_r],$$

$$F_2 : K_2 \to B_2, \quad \Psi_{r,s} \mapsto [g_r, g_s] - [g_s, g_r], \quad \Psi_{r,r} \mapsto \sum_{l=0}^{m_r-1} [g_r^l, g_r],$$

and

$$F_3 : K_3 \to B_3,$$
$$\Psi_{r,s,t} \mapsto [g_r, g_s, g_t] - [g_s, g_r, g_t] - [g_r, g_t, g_s] + [g_t, g_r, g_s] + [g_s, g_t, g_r] - [g_t, g_s, g_r],$$
$$\Psi_{r,r,s} \mapsto \sum_{l=0}^{m_r-1} \left([g_r^l, g_r, g_s] - [g_r^l, g_s, g_r] + [g_s, g_r^l, g_r] \right),$$
$$\Psi_{r,s,s} \mapsto \sum_{l=0}^{m_s-1} \left([g_r, g_s^l, g_s] - [g_s^l, g_r, g_s] + [g_s^l, g_s, g_r] \right),$$
$$\Psi_{r,r,r} \mapsto \sum_{l=0}^{m_r-1} [g_r, g_r^l, g_r],$$

for $0 \le r \le k$, $0 \le r < s \le k$ and $0 \le r < s < t \le k$.

Lemma 3.9. *The following diagram is commutative:*

$$
\begin{array}{ccccccccccc}
\cdots & \xrightarrow{} & K_3 & \xrightarrow{d} & K_2 & \xrightarrow{d} & K_1 & \xrightarrow{d} & K_0 & \longrightarrow & \mathbb{Z} & \longrightarrow 0 \\
& & \downarrow{\scriptstyle F_3} & & \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_1} & & \| & & \| & \\
\cdots & \xrightarrow{} & B_3 & \xrightarrow{\partial_3} & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\partial_1} & B_0 & \longrightarrow & \mathbb{Z} & \longrightarrow 0.
\end{array}
$$

Proof. The proof is routine and indeed becomes much easier, so we omit it. □

Corollary 3.10. *Let $\phi \in B_3^*$ be a 3-cocycle. Then ϕ is a 3-coboundary if and only if $F_3^*(\phi)$ is a 3-coboundary.*

Proof. Follows from the fact that F_3^* induces an isomorphism between 3-cohomology groups. □

3.3. Abelian cocycles. We start with the definition of abelian cocycles. For this, we need to recall the definition of the twisted quantum double. The *twisted quantum double* $D^\Phi(G)$ of G with respect to the 3-cocycle Φ over G is the semisimple quasi-Hopf algebra with underlying vector space $(\mathbf{k}G)^* \otimes \mathbf{k}G$ in which multiplication, comultiplication Δ, associator ϕ, counit ε, antipode $\mathcal{S}$, α and β are given by

$$(e(g) \otimes x)(e(h) \otimes y) = \theta_g(x, y)\delta_{g^x,h} \, e(g) \otimes xy,$$
$$\Delta(e(g) \otimes x) = \sum_{hk=g} \gamma_x(h, k)e(h) \otimes x \otimes e(k) \otimes x,$$
$$\phi = \sum_{g,h,k \in G} \Phi(g, h, k)^{-1} e(g) \otimes 1 \otimes e(h) \otimes 1 \otimes e(k) \otimes 1,$$
$$\mathcal{S}(e(g) \otimes x) = \theta_{g^{-1}}(x, x^{-1})^{-1}\gamma_x(g, g^{-1})^{-1} e(x^{-1}g^{-1}x) \otimes x^{-1},$$
$$\varepsilon(e(g) \otimes x) = \delta_{g,1}, \quad \alpha = 1, \quad \beta = \sum_{g \in G} \Phi(g, g^{-1}, g)e(g) \otimes 1,$$

where $\{e(g) \mid g \in G\}$ is the dual basis of $\{g \mid g \in G\}$, $\delta_{g,1}$ is the Kronecker delta, $g^x = x^{-1}gx$, and

$$\theta_g(x, y) = \frac{\Phi(g, x, y)\Phi(x, y, (xy)^{-1}gxy)}{\Phi(x, x^{-1}gx, y)},$$

$$\gamma_g(x, y) = \frac{\Phi(x, y, g)\Phi(g, g^{-1}xg, g^{-1}yg)}{\Phi(x, g, g^{-1}yg)}$$

for any $x, y, g \in G$ (cf. [8]).

Clearly, M is a left $D^\Phi(G)$-module if and only if M is a left-left Yetter–Drinfeld module over $(\mathbf{k}G, \Phi)$ as defined in the previous section. For our purpose, we prefer the following equivalent definition of abelian cocycles via twisted quantum doubles appeared in [28].

Definition 3.11. A 3-cocycle Φ over G is called *abelian* if $D^\Phi(G)$ is a commutative algebra.

Remark 3.12. Abelian cocycles of the previous form and some related properties were discussed by Ng [29] and by Mason–Ng [28]. In [29], Ng gave a quite symmetric description of abelian cocycles. Note that the Eilenberg–MacLane abelian cocycles [10] are different from the present ones. Recall that, an Eilenberg–MacLane abelian cocycle is a pair (Φ, d) where $\Phi \in \mathbb{Z}^3(G, \mathbf{k}^*)$ and d is a braiding which is compatible with Φ. But, we still have the following observation: if (Φ, d) is an Eilenberg–MacLane abelian cocycle, then Φ must be an abelian cocycle in our sense. As this fact is not necessary for our following discussions, here we won't provide a proof.

As a direct consequence of this definition, we have the following conclusion.

Corollary 3.13. *Every Yetter–Drinfeld module over* $(\mathbf{k}G, \Phi)$ *is diagonal if and only if* Φ *is abelian.*

Now we go back to the situation where $\mathbb{G}$ is an abelian group. So $\mathbb{G} \cong \mathbb{Z}_{\mathrm{m}_1} \times \cdots \times \mathbb{Z}_{\mathrm{m}_n}$ with $\mathrm{m}_j \in \mathbb{N}$ for $1 \leq j \leq n$ and $\mathrm{m}_i \mid \mathrm{m}_{i+1}$ for all $1 \leq i \leq n-1$. Let g_i be a generator of $\mathbb{Z}_{\mathrm{m}_i}$. By Proposition 3.8, we can assume that $\Phi = \Phi_{\underline{a}}$ for some $\underline{a} \in A$. Using our formula of 3-cocycles, we have the following conclusion which provides a quite explicit description of abelian cocycles.

Proposition 3.14. *The* 3-cocycle $\Phi_{\underline{a}}$ *is abelian if and only if*

$$a_{rst} = 0$$

for all $1 \leq r < s < t \leq n$.

Proof. "$\Leftarrow$" If all $a_{rst} = 0$, then by (3.10) it is not hard to find that

$$\Phi_{\underline{a}}(x, y, z) = \Phi_{\underline{a}}(x, z, y)$$

for $x, y, z \in \mathbb{G}$. From this, we can find that

$$\theta_g(x, y) = \theta_g(y, x)$$

for $g, x, y \in \mathbb{G}$, which implies that $D^{\Phi_{\underline{a}}}(\mathbb{G})$ is a commutative.

"⇒" If $a_{rst} \neq 0$ for some $r < s < t$. For the sake of simplicity, assume that $a_{123} \neq 0$. Through direct computations, we have

$$\theta_{g_1}(g_2, g_3) = 1, \quad \theta_{g_1}(g_3, g_2) = \zeta_{m_1}^{-a_{123}}.$$

This implies that

$$(e(g_1) \otimes g_2)(e(g_1) \otimes g_3) \neq (e(g_1) \otimes g_3)(e(g_1) \otimes g_2). \qquad \square$$

3.4. Resolution. Let $\mathbb{G} = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ be as before and let $\Phi_{\underline{a}}$ be an abelian 3-cocycle of $\mathbb{G}$. One of our key observations is that $\Phi_{\underline{a}}$ can be "resolved" in a slightly bigger abelian group G. More precisely, take $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ for $m_i = m_i^2$ $(1 \leq i \leq n)$. As before, let g_i (resp. g_i) be a generator of $\mathbb{Z}_{m_i}$ (resp. $\mathbb{Z}_{m_i}$) for $1 \leq i \leq n$. Using such notations, we have a canonical group epimorphism:

$$\pi : G \to \mathbb{G}, \quad g_i \mapsto g_i \quad (1 \leq i \leq n).$$

From this map, we can pull back the 3-cocycles of $\mathbb{G}$ and get many 3-cocycles over G. That is, the map

$$\pi^*(\Phi_{\underline{a}}) : G \times G \times G \to k^*, \quad (g, h, z) \mapsto \Phi_{\underline{a}}(\pi(g), \pi(h), \pi(z)), \quad g, h, z \in G$$

is a 3-cocycle of G. Our observation is that $\pi^*(\Phi_{\underline{a}})$ is indeed a boundary. In fact, consider the map

$$(3.11) \qquad J_{\underline{a}} : G \times G \to k^*,$$

$$(g_1^{x_1} \cdots g_n^{x_n}, g_1^{y_1} \cdots g_n^{y_n}) \mapsto \prod_{l=1}^n \zeta_{m_l}^{a_l x_l (y_l - y_l')} \prod_{1 \leq s < t \leq n} \zeta_{m_s m_t}^{a_{st} x_t (y_s - y_s')},$$

where y_i' is the remainder of y_i divided by m_i for $1 \leq i \leq n$. For simplicity, we just take $\zeta_t = e^{2\pi i/t}$ for $t \in \mathbb{N}$. We are thus led to the following result:

Proposition 3.15. *The differential of $J_{\underline{a}}$ equals $\pi^*(\Phi_{\underline{a}})$, that is,*

$$\partial(J_{\underline{a}}) = \pi^*(\Phi_{\underline{a}}).$$

Proof. Indeed,

$$\partial(J_{\underline{a}})(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n})$$

$$= \frac{J_{\underline{a}}(g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}) J_{\underline{a}}(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1+k_1} \cdots g_n^{j_n+k_n})}{J_{\underline{a}}(g_1^{i_1+j_1} \cdots g_n^{i_n+j_n}, g_1^{k_1} \cdots g_n^{k_n}) J_{\underline{a}}(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n})}$$

$$= \left(\prod_{l=1}^n \zeta_{m_l}^{a_l j_l (k_l - k_l')} \prod_{1 \leq s < t \leq n} \zeta_{m_s m_t}^{a_{st} j_t (k_s - k_s')} \prod_{l=1}^n \zeta_{m_l}^{a_l i_l (j_l + k_l - (j_l + k_l)')} \right.$$

$$\times \prod_{1 \leq s < t \leq n} \zeta_{m_s m_t}^{a_{st} i_t (j_s + k_s - (j_s + k_s)')} \left(\prod_{l=1}^n \zeta_{m_l}^{a_l (i_l + j_l)(k_l - k_l')} \right.$$

$$\times \left. \prod_{1 \leq s < t \leq n} \zeta_{m_s m_t}^{a_{st} (i_t + j_t)(k_s - k_s')} \prod_{l=1}^n \zeta_{m_l}^{a_l i_l (j_l - j_l')} \prod_{1 \leq s < t \leq n} \zeta_{m_s m_t}^{a_{st} i_t (j_s - j_s')} \right)^{-1}$$

$$= \prod_{l=1}^{n} \zeta_{\mathfrak{m}_l}^{a_l i_l (j'_l + k'_l - (j_l + k_l)')} \prod_{1 \le s < t \le n} \zeta_{\mathfrak{m}_s \mathfrak{m}_t}^{a_{st} i_t (j'_s + k'_s - (j_s + k_s)')}$$

$$= \prod_{l=1}^{n} \zeta_{\mathfrak{m}_l}^{a_l i'_l [\frac{j'_l + k'_l}{\mathfrak{m}_l}]} \prod_{1 \le s < t \le n} \zeta_{\mathfrak{m}_t}^{a_{st} i'_t [\frac{j'_s + k'_s}{\mathfrak{m}_s}]}$$

$$= \pi^*(\Phi_{\underline{a}})(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}). \qquad \square$$

Although the above conclusion is true for abelian 3-cocycles, it does not hold for non-abelian 3-cocycles. Precisely, let Φ be a non-abelian 3-cocycle on $\mathbb{G}$. Then we will show that there does not exist any finite abelian group G' such that there is a group epimorphism $\pi : G' \to \mathbb{G}$ making $\pi^*(\Phi)$ to be a coboundary (this is a surprising phenomenon, at least to us). To prove this fact, we start with the following special case, and then reduce the general case to this special case.

Lemma 3.16. *Let $\Phi_{\underline{a}}$ be a non-abelian 3-cocycle on $\mathbb{G}$. Suppose that we have a group epimorphism*

$$\pi : \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_n} = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle \to \mathbb{G} = \langle \mathbb{g}_1 \rangle \times \cdots \times \langle \mathbb{g}_n \rangle, \quad g_i \mapsto \mathbb{g}_i.$$

Then $\pi^*(\Phi_{\underline{a}})$ is not a coboundary on $\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_n}$.

Proof. Since $\Phi_{\underline{a}}$ is not an abelian cocycle, there are $r < s < t$ such that $a_{rst} \neq 0$ by Proposition 3.14. Without loss of generality, we assume that $a_{123} \neq 0$. Assume that $\pi^*(\Phi_{\underline{a}})$ is a coboundary. By Corollary 3.10, $F_3^*(\pi^*(\Phi_{\underline{a}}))$ is coboundary and then Lemma 3.3 implies that $F_3^*(\pi^*(\Phi_{\underline{a}}))_{1,2,3} = 1$. But, direct computation shows that

$$\begin{aligned} F_3^*(\pi^*(\Phi_{\underline{a}}))_{1,2,3} &= F_3^*(\pi^*(\Phi_{\underline{a}}))(\Psi_{1,2,3}) \\ &= \pi^*(\Phi_{\underline{a}})\big([g_1, g_2, g_3] - [g_2, g_1, g_3] - [g_1, g_3, g_2] \\ &\quad + [g_3, g_1, g_2] + [g_2, g_3, g_1] - [g_3, g_2, g_1]\big) \\ &= \Phi_{\underline{a}}\big([\mathbb{g}_1, \mathbb{g}_2, \mathbb{g}_3] - [\mathbb{g}_2, \mathbb{g}_1, \mathbb{g}_3] - [\mathbb{g}_1, \mathbb{g}_3, \mathbb{g}_2] \\ &\quad + [\mathbb{g}_3, \mathbb{g}_1, \mathbb{g}_2] + [\mathbb{g}_2, \mathbb{g}_3, \mathbb{g}_1] - [\mathbb{g}_3, \mathbb{g}_2, \mathbb{g}_1]\big) \\ &= \zeta_{\mathfrak{m}_1}^{a_{123}} \neq 1. \end{aligned}$$

This is a direct contradiction. $\qquad \square$

Proposition 3.17. *Let $\Phi_{\underline{a}}$ be a non-abelian 3-cocycle on $\mathbb{G}$ and let G be an arbitrary finite abelian group. Suppose that we have a group epimorphism $\pi : G \twoheadrightarrow \mathbb{G}$. Then $\pi^*(\Phi_{\underline{a}})$ is not a coboundary on G.*

Proof. On the contrary, assume that $\pi^*(\Phi_{\underline{a}})$ is a coboundary on G. Let g_i be a preimage of $\mathbb{g}_i$ for $1 \le i \le n$. Let G_1 be the subgroup generated by $g_1, \ldots, g_n$ and so we have a group embedding $\iota : G_1 \to G$. Assume that $\mathrm{ord}(g_i) = l_i$. Then clearly we have the following group epimorphism:

$$\pi' : \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_n} = \langle h_1 \rangle \times \cdots \times \langle h_n \rangle \twoheadrightarrow G_1, \quad h_i \mapsto g_i.$$

Define $f_1 := \iota \circ \pi' : \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_n} \to G$ and $f := \pi \circ f_1$. Note that by definition, the map f is given by

$$\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_n} \to \mathbb{G}, \quad g_i \mapsto \mathbb{g}_i \quad (1 \le i \le n).$$

If $\pi^*(\Phi_{\underline{a}})$ is a coboundary on G, then $f_1^*(\pi^*(\Phi_{\underline{a}})) = f^*(\Phi_{\underline{a}})$ is still a coboundary. But this is absurd by Lemma 3.16. $\qquad\square$

4. Nichols algebras of diagonal type in $\mathop{}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$

The aim of this section is to give a classification of the Nichols algebras of diagonal type with arithmetic root system in $\mathop{}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. The idea to realize our purpose consists of five steps. Firstly, we can assume that the support group of $\mathcal{B}(V)$ is $\mathbb{G}$, and from this assumption we can prove that Φ must be an abelian 3-cocycle over $\mathbb{G}$. Secondly, we will develop a technique to change the base group from $\mathbb{G}$ to a bigger one G together with a group epimorphism $\pi : G \to \mathbb{G}$. Thirdly, we will show that any Nichols algebra $\mathcal{B}(V)$ in $\mathop{}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$ is isomorphic to a Nichols algebra in $\mathop{}_{G}^{G}\mathcal{YD}^{\pi^*(\Phi)}$, which is thus twist equivalent to a usual Nichols algebra by Proposition 3.15. Fourthly, we want to get a return ticket, that is, we will give a sufficient and necessary condition to determine when a Nichols algebra in $\mathop{}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi)}$ is isomorphic to one in $\mathop{}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. Finally, combining these results and Heckenberger's classification of arithmetic root systems, we obtain the classification of Nichols algebras of diagonal type with arithmetic root system in $\mathop{}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$.

4.1. Start points. We give two conclusions as our preparations for classification. At first, we will prove that any Nichols algebras of diagonal type can be realized in $\mathop{}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$, where $\mathbb{G}$ is an abelian group and Φ is an abelian 3-cocycle over $\mathbb{G}$. Recall that in Remark 2.8, we gave the definition of the support group for any $V \in \mathop{}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. For convenience, we denote the support of V by G_V and it is not hard to see that $G_V = G_{\mathcal{B}(V)} = G_{T_{\Phi}(V)}$.

Lemma 4.1. *Let $\mathcal{B}(V)$ be a Nichols algebra of diagonal type in $\mathop{}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$, and let $\mathbb{G}' = G_{\mathcal{B}(V)}$ be the support group of $\mathcal{B}(V)$. Let $\Psi = \Phi|_{\mathbb{G}'}$. Then $\mathbb{G}'$ is an abelian group and Ψ is an abelian 3-cocycle over $\mathbb{G}'$.*

Proof. Firstly by Remark 2.8, we know that $\mathbb{G}'$ lies in the center of $\mathbb{G}$ and thus it is an abelian group.

Next we will prove that Ψ is an abelian 3-cocycle over $\mathbb{G}'$. Assume without loss of generality that $\mathbb{G}' = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle$, and Ψ is of the form (3.10). According to Proposition 3.14, we only need to prove that $a_{rst} = 0$ for all $1 \le r < s < t \le n$.

At first, fix a triple (r, s, t) such that $1 \le r < s < t \le n$. Since $\mathbb{G}' = \langle h_1, \ldots, h_m \rangle$, we have $g_r = h_1^{k_1} \cdots h_m^{k_m}$, where $k_1 < |h_1|, \ldots, k_m < |h_m|$. Here $|g|$ means the order of g. Conversely, h_i $(1 \le i \le m)$ can be presented by the generators of $\mathbb{G}'$, i.e. $h_i = g_1^{c_{i1}} \cdots g_n^{c_{in}}$, and we get

$$\sum_{i=1}^{m} k_i c_{il} \equiv \begin{cases} 0 \ (\mathrm{mod}\, m_l) & \text{if } l \ne r, \\ 1 \ (\mathrm{mod}\, m_r) & \text{if } l = r. \end{cases}$$

By (3.10), we have

$$\Psi(h_i, g_s, g_t) = \prod_{1 \leq j < s} \zeta_{(m_j, m_s, m_t)}^{c_{ij} a_{jst}},$$

so

$$\prod_{i=1}^{m} \Psi(h_i, g_s, g_t)^{k_i} = \prod_{1 \leq j < s} \zeta_{(m_j, m_s, m_t)}^{a_{jst}(\sum_{i=1}^{n} k_i c_{ij})} = \zeta_{(m_r, m_s, m_t)}^{a_{rst}}.$$

On the other hand, since $\mathbf{k} X_i$ $(1 \leq i \leq m)$ are one-dimensional $(\mathbf{k} \mathbb{G}', \widetilde{\Psi}_{h_i})$-representations, $\widetilde{\Psi}_{h_i}(g_s, g_t) = \widetilde{\Psi}_{h_i}(g_t, g_s)$. It follows by a direct computation that $\widetilde{\Psi}_{h_i}(g_s, g_t) = 1$ (since $s < t$), and thus

$$\widetilde{\Psi}_{h_i}(g_t, g_s) = \widetilde{\Psi}_{h_i}(g_s, g_t) = 1.$$

Hence

$$\Psi(h_i, g_s, g_t) = \frac{\Psi(g_s, h_i, g_t)}{\Psi(g_s, g_t, h_i)} = \prod_{s < p < t} \zeta_{(m_s, m_p, m_t)}^{c_{ip} a_{spt}} \left[\prod_{t < q \leq n} \zeta_{(m_s, m_t, m_q)}^{c_{iq} a_{stq}} \right]^{-1}.$$

Then we get

$$\prod_{i=1}^{m} \Psi(h_i, g_s, g_t)^{k_i} = \prod_{i=1}^{m} \left\{ \prod_{s < p < t} \zeta_{(m_s, m_p, m_t)}^{c_{ip} a_{spt}} \left[\prod_{t < q \leq n} \zeta_{(m_s, m_t, m_q)}^{c_{iq} a_{stq}} \right]^{-1} \right\}^{k_i}$$

$$= \prod_{s < p < t} \zeta_{(m_s, m_p, m_t)}^{a_{spt}(\sum_{i=1}^{m} k_i c_{ip})} \left[\prod_{t < q \leq n} \zeta_{(m_s, m_t, m_q)}^{a_{stq}(\sum_{i=1}^{m} k_i c_{iq})} \right]^{-1}$$

$$= 1.$$

So we obtain $\zeta_{(m_r, m_s, m_t)}^{a_{rst}} = 1$ and this implies $a_{rst} = 0$ since $0 \leq a_{rst} < (m_r, m_s, m_t)$.   □

For our purpose of classification of Nichols algebras $\mathcal{B}(V)$ of diagonal type in ${}_{G}^{G}\mathcal{YD}^{\Phi}$, it is harmless to *assume in the rest of the paper that the support group of $\mathcal{B}(V)$ is G and thus G is abelian and $\Phi = \Phi_{\underline{a}}$ is an abelian cocycle.*

Secondly, we will show that there is a nice grading on $\mathcal{B}(V) \in {}_{G}^{G}\mathcal{YD}^{\Phi}$. Let $V \in {}_{G}^{G}\mathcal{YD}^{\Phi}$ be a Yetter–Drinfeld module of diagonal type and $\{X_i \mid 1 \leq i \leq l\}$ a canonical basis of V. Let $\mathbb{Z}^l$ be the free abelian group of rank l and assume that e_i $(1 \leq i \leq l)$ are the canonical generators of $\mathbb{Z}^l$. The following fact is very important for our follow-up discussions, which is indeed [23, Lemma 4.2]. We include a proof here for completeness and safety.

Proposition 4.2. *There is a $\mathbb{Z}^l$-grading on the Nichols algebra $\mathcal{B}(V) \in {}_{G}^{G}\mathcal{YD}^{\Phi}$ by setting* $\deg X_i = e_i$.

Proof. Obviously, there is a $\mathbb{Z}^l$-grading on the tensor algebra $T_{\Phi}(V) \in {}_{G}^{G}\mathcal{YD}^{\Phi}$ by assigning $\deg X_i = e_i$. Let $I = \bigoplus_{i \geq 0} I_i$ be the maximal graded Hopf ideal generated by homogeneous elements of degree greater than or equal to 2. To prove that $\mathcal{B}(V)$ is $\mathbb{Z}^l$-graded, it amounts to prove that I is $\mathbb{Z}^l$-graded. This will be done by induction on the $\mathbb{N}$-degree.

To this aim, let $I^k := \bigoplus_{0 \leq i \leq k} I_i$ for $k \geq 0$. Since $I = \bigoplus_{i \geq 0} I_i$ is generated by homogeneous elements of degree greater than or equal to 2, we have

$$I^0 = I_0 = 0 \quad \text{and} \quad I^1 = I_0 \oplus I_1 = 0.$$

Hence I^0 and I^1 are $\mathbb{Z}^l$-graded. Now suppose I^k is $\mathbb{Z}^l$-graded for a fixed $k \geq 1$. We shall prove that $I^{k+1} = \bigoplus_{0 \leq i \leq k+1} I_i$ is also $\mathbb{Z}^l$-graded.

Let $X \in I_{k+1}$ and $X = X^1 + X^2 + \cdots + X^n$, with each X^i being $\mathbb{Z}^l$-homogeneous and X^i and X^j having different $\mathbb{Z}^l$-degrees if $i \neq j$. Write

$$\Delta(X^i) = X^i \otimes 1 + 1 \otimes X^i + (X^i)_1 \otimes (X^i)_2.$$

By definition, we have $\Delta(X) = X \otimes 1 + 1 \otimes X + (X)_1 \otimes (X)_2$, where

$$(X)_1 \otimes (X)_2 \in T_\Phi(V) \otimes I^k + I^k \otimes T_\Phi(V),$$

i.e., $\sum (X^i)_1 \otimes (X^i)_2 \in T_\Phi(V) \otimes I^k + I^k \otimes T_\Phi(V)$. According to the inductive assumption, $T_\Phi(V) \otimes I^k + I^k \otimes T_\Phi(V)$ is a $\mathbb{Z}^l$-graded space. So each $(X^i)_1 \otimes (X^i)_2$ is an element of $T_\Phi(V) \otimes I^k + I^k \otimes T_\Phi(V)$ as Δ preserves $\mathbb{Z}^l$-degrees. If there is an $X^i \notin I_{k+1}$, then $I + \langle X^i \rangle$ is a Hopf ideal properly containing I, which contradicts the maximality of I. It follows that $X^i \in I_{k+1}$ for all $1 \leq i \leq n$ and hence I^{k+1} is also $\mathbb{Z}^l$-graded by the assumption on X. This completes the proof of the proposition. $\qquad\square$

4.2. Change of base groups. Since Nichols algebras in the braided tensor category ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^\Phi$ are non-associative algebras, the structures of these algebras depend on $\mathbb{G}$ and the 3-cocycle Φ on $\mathbb{G}$. We will call $\mathbb{G}$ the base group of $\mathcal{B}(V)$. One of the most important methods of this paper is to change the base groups of Nichols algebras. We need the following definition.

Definition 4.3. Let $\mathcal{B}(V)$ and $\mathcal{B}(U)$ be Nichols algebras in ${}^{G}_{G}\mathcal{YD}^\Phi$ and ${}^{H}_{H}\mathcal{YD}^\Psi$, respectively, with $\dim V = \dim U = l$. We say $\mathcal{B}(V)$ is isomorphic to $\mathcal{B}(U)$ if there is a $\mathbb{Z}^l$-graded linear isomorphism $\mathcal{F} : \mathcal{B}(V) \to \mathcal{B}(U)$ which preserves multiplication and comultiplication.

Lemma 4.4. *Suppose* $(V, \delta_V, \rhd) \in {}^{G}_{G}\mathcal{YD}^\Phi$ *and* $(U, \delta_U, \rhd) \in {}^{H}_{H}\mathcal{YD}^\Psi$. *Let* G' *and* H' *be support groups of* V *and* U, *respectively. If there are a linear isomorphism* $F : V \to U$ *and a group epimorphism* $f : G' \to H'$ *such that*

$$\tag{4.1} \delta_U \circ F = (f \otimes F) \circ \delta_V,$$

$$\tag{4.2} F(g \rhd v) = f(g) \rhd F(v),$$

$$\Phi|_{G'} = f^* \Psi|_{H'}$$

for any $g \in G'$, $v \in V$, *then* $\mathcal{B}(V)$ *is isomorphic to* $\mathcal{B}(U)$.

Proof. Let $\mathcal{F} : T_{\Phi|_{G'}}(V) \to T_{f^*(\Psi|_{H'})}(U)$ be the multiplicative linear map such that $\mathcal{F}|_V = F$. It is easy to show that F also preserves the comultiplication between $T_{\Phi|_{G'}}(V)$ and $T_{f^*(\Psi|_{H'})}(U)$. By Corollary 3.13 and Lemma 4.1, both V and U have canonical basis. Let $\{X_i \mid 1 \leq i \leq l\}$ be a canonical basis of V, then it is obvious that $\{Y_i = F(X_i) \mid 1 \leq i \leq l\}$ is a canonical basis of U by (4.1) and (4.2). Let $\{e_i \mid 1 \leq i \leq l\}$ be the free generators of $\mathbb{Z}^l$. Then $T_{\Phi|_{G'}}(V)$ and $T_{f^*(\Psi|_{H'})}(U)$ are $\mathbb{Z}^l$-graded by setting $\deg(X_i) = \deg(Y_i) = e_i$. Note that $\mathcal{F}$ induces a one-to-one correspondence between the set of $\mathbb{Z}^l$-graded Hopf ideals of $T_{\Phi|_{G'}}(V)$ and that of $T_{f^*(\Psi|_{H'})}(U)$. By Proposition 4.2, we know that the maximal Hopf ideals generated by homogeneous elements of degree ≥ 2 in $T_{\Phi|_{G'}}(V)$ and in $T_{f^*(\Psi|_{H'})}(U)$ are $\mathbb{Z}^l$-graded. It is obvious that $\mathcal{F}$ maps the maximal Hopf ideal of $T_{\Phi|_{G'}}(V)$ to that of $T_{f^*(\Psi|_{H'})}(U)$. Therefore, $\mathcal{F}$ induces a linear isomorphism from $\mathcal{B}(V)$ to $\mathcal{B}(U)$ which preserves multiplication and comultiplication. $\qquad\square$

The following definition is convenient for our later expositions.

Definition 4.5.   If (F, f) is an isomorphism from $\mathcal{B}(V)$ to $\mathcal{B}(U)$ as in Lemma 4.4, then we say $\mathcal{B}(V)$ is isomorphic to $\mathcal{B}(U)$ *through the group morphism* f.

Suppose

$$
\mathbb{G} = \mathbb{Z}_{\mathrm{m}_1} \times \cdots \times \mathbb{Z}_{\mathrm{m}_n} = \langle \mathrm{g}_1 \rangle \times \cdots \times \langle \mathrm{g}_n \rangle,
$$
$$
G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle,
$$

where $m_i = \mathrm{m}_i^2$ for $1 \leq i \leq n$. Let

$$
\pi : \mathbf{k}G \to \mathbf{k}\mathbb{G}, \quad g_i \mapsto \mathrm{g}_i, \quad 1 \leq i \leq n
$$

be the canonical epimorphism. Observe that π has a section

$$
\iota : \mathbf{k}\mathbb{G} \to \mathbf{k}G, \quad \prod_{i=1}^{n} \mathrm{g}_i^{i_j} \mapsto \prod_{i=1}^{n} g_i^{i_j}
$$

which is not a group morphism. Let δ_L and $\triangleright$ be the comodule and module structure maps of $V \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. Define

$$
\rho_L : V \to \mathbf{k}G \otimes V, \qquad \rho_L = (\iota \otimes \mathrm{id})\delta_L,
$$
$$
\blacktriangleright : \mathbf{k}G \otimes V \to V, \quad g \blacktriangleright Z = \pi(g) \triangleright Z
$$

for all $g \in G$ and $Z \in V$.

Lemma 4.6.   *Defined in this way,* $(V, \rho_L, \blacktriangleright)$, *denoted simply by* $\widetilde{V}$ *in the following, is an object in* ${}^{G}_{G}\mathcal{YD}^{\pi^*\Phi}$.

Proof.   This can be verified by direct computation:

$$
\begin{aligned}
e \blacktriangleright (f \blacktriangleright Z) &= \pi(e) \triangleright (\pi(f) \triangleright Z) \\
&= \tilde{\Phi}_z(\pi(e), \pi(f))(\pi(e)\pi(f)) \triangleright Z \\
&= \widetilde{\pi^*(\Phi)}_{\iota(z)}(e, f)ef \blacktriangleright Z
\end{aligned}
$$

for all $e, f \in G$ and $\delta_L(Z) = z \otimes Z$ for $Z \in V$. $\square$

Proposition 4.7.   *For any Nichols algebra* $\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$, *the Nichols algebra* $\mathcal{B}(\widetilde{V}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi)}$ *is isomorphic to* $\mathcal{B}(V)$. *Moreover,* $\mathcal{B}(\widetilde{V})$ *is twist equivalent to a usual Nichols algebra.*

Proof.   The first statement is a direct consequence of Lemma 4.4. For the second, just note that $\pi^*(\Phi)$ is a 3-coboundary on G by Proposition 3.15. $\square$

To summarize so far, we have found the following route of transforming a *non-associative* Nichols algebra to a *usual* one:

(4.3) $\qquad\qquad \mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ (original Nichols algebra)

$$\downarrow \text{Lemma 4.4 and Lemma 4.6}$$

$$\mathcal{B}(V) \cong \mathcal{B}(\widetilde{V}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi)}$$

$$\downarrow \text{Proposition 3.15}$$

$\mathcal{B}(\widetilde{V})$ is twisted equivalent to a usual Nichols algebra $\mathcal{B}(V)'$

Since we only want to classify finite-dimensional Majid algebras, there is no harm to assume that all the usual Nichols algebras appearing in this paper have arithmetic root systems. According to this diagram, each diagonal Nichols algebra $\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ is corresponding to a usual diagonal Nichols algebra, denoted by $\mathcal{B}(V)'$ for convenience, in the above way (and thus a unique way). Note that there is a $\mathbb{Z}^l$-graded linear isomorphism $\mathcal{B}(V) \cong \mathcal{B}(V)'$. Thus, it is reasonable to make the following definition.

Definition 4.8. The arithmetic root system of $\mathcal{B}(V)$ is defined to be that of $\mathcal{B}(V)'$. That is, $\triangle(\mathcal{B}(V))_{\chi,E} := \triangle(\mathcal{B}(V)')_{\chi,E}$ by the prescribed notations in Section 2.5. In particular, the root system $\triangle(\mathcal{B}(V))$ of $\mathcal{B}(V)$ equals $\triangle(\mathcal{B}(V)')$.

The aim of this section is to classify the Nichols algebras of diagonal type with arithmetic root system in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. Using diagram (4.3), we just need to answer the following question: For a usual Nichols algebra $\mathcal{B}$ of diagonal type with arithmetic root system, when is $\mathcal{B}$ gotten from a Nichols algebra $\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$? That is, find a return trip of diagram (4.3).

4.3. The return trip. Keep the notations of the previous subsection. At first, we give the inverse version of Proposition 4.7.

Lemma 4.9. *Let $\mathcal{B}(\widetilde{V}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi)}$ be a Nichols algebra of diagonal type and let $\{Y_i \mid 1 \le i \le m\}$ be a canonical basis of $\widetilde{V}$. Then $\mathcal{B}(\widetilde{V})$ is isomorphic to a Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ through π if and only if*

$$g_i^{\mathrm{m}_i} \blacktriangleright Y_j = Y_j, \quad 1 \le i \le n, 1 \le j \le m.$$

Proof. If $g_i^{\mathrm{m}_i} \blacktriangleright Y_j = Y_j$, then one can easily show $\widetilde{V}$ is an object in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ by defining

$$\delta_L : V \to \mathbf{k}\mathbb{G} \otimes V, \qquad \delta_L = (\pi \otimes \mathrm{id})\rho_L,$$
$$\triangleright : \mathbf{k}\mathbb{G} \otimes V \to V, \quad \mathrm{g} \triangleright Z = \iota(\mathrm{g}) \blacktriangleright Z.$$

It is obvious that (id_V, π) is an isomorphism between the Nichols algebras $\mathcal{B}(\widetilde{V}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi)}$ and $\mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$.

For the other direction, suppose (F, π) is an isomorphism from $\mathcal{B}(V) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi)}$ to $\mathcal{B}(U) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. Then by Definition 4.8 and equation (4.2), we have

$$F(g_i^{\mathrm{m}_i} \blacktriangleright Y_j) = \pi(g_i^{\mathrm{m}_i}) \blacktriangleright F(Y_j) = F(Y_j).$$

This implies $g_i^{\mathrm{m}_i} \blacktriangleright Y_j = Y_j$ for all $1 \le i \le n, 1 \le j \le m$. $\qquad\square$

Now fix a usual Nichols algebra of diagonal type $\mathcal{B}(V)' \in {}_G^G\mathcal{YD}$ with support group G. According to (4.3), we need to answer the following question:

Question 4.10. When is $\mathcal{B}(V)'^{J_{\underline{a}}}$ isomorphic to a Nichols algebra in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$ through π?

Let $\{X_i \mid 1 \leq i \leq m\}$ be a canonical basis of V. Assume that

$$\delta_L'(X_i) = h_i \otimes X_i, \quad g_k \rhd' X_j = q_{kj} X_j$$

for $1 \leq i, j \leq m, 1 \leq k \leq n, h_i \in G$ and $q_{kj} \in \mathbf{k}^*$, where δ_L' (resp. $\rhd'$) is the comodule (resp. module) structure map of $\mathcal{B}(V)' \in {}_G^G\mathcal{YD}$. So there are $0 \leq x_{kj}, s_{ik} < m_k$ such that

$$q_{kj} = \zeta_{m_k}^{x_{kj}}, \quad h_i = \prod_{k=1}^{n} g_k^{s_{ik}}$$

for $1 \leq i, j \leq m$ and $1 \leq k \leq n$. Let $\mathbf{X} = (x_{ij})_{n \times m}$. By assumption, the support group $G_{\mathcal{B}(V)'} = G$ and $\{h_i \mid 1 \leq i \leq m\}$ generate the group G. So there are $t_{jl} \in \mathbb{N}$ such that

$$g_j = \prod_{l=1}^{m} h_l^{t_{jl}}, \quad 1 \leq j \leq n.$$

By $\mathbf{S}$ and $\mathbf{T}$, we denote the matrices $(s_{ik})_{m \times n}$ and $(t_{jl})_{n \times m}$. It is obvious that

$$(4.4) \qquad \mathbf{TS} \equiv \begin{pmatrix} 1 \ (\mathrm{mod}\, m_1) & 0 \ (\mathrm{mod}\, m_1) & \cdots & 0 \ (\mathrm{mod}\, m_1) \\ 0 \ (\mathrm{mod}\, m_2) & 1 \ (\mathrm{mod}\, m_2) & \cdots & 0 \ (\mathrm{mod}\, m_2) \\ \vdots & \vdots & & \vdots \\ 0 \ (\mathrm{mod}\, m_n) & 0 \ (\mathrm{mod}\, m_n) & \cdots & 1 \ (\mathrm{mod}\, m_n) \end{pmatrix}.$$

With these notations, we can now give the answer to Question 4.10.

Proposition 4.11. *The twisting $\mathcal{B}(V)'^{J_{\underline{a}}}$ is isomorphic to a Nichols algebra in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$ through π if and only if the following congruence equalities hold:*

$$(4.5) \qquad \sum_{j=1}^{m} x_{ij} t_{lj} \equiv 0 \ (\mathrm{mod}\ \mathrm{m}_i), \quad 1 \leq l < i \leq n,$$

$$(4.6) \qquad \sum_{j=1}^{m} x_{ij} t_{ij} \equiv a_i \ (\mathrm{mod}\ \mathrm{m}_i), \quad 1 \leq i \leq n,$$

$$(4.7) \qquad \left(\sum_{j=1}^{m} x_{ij} t_{lj}\right) \mathrm{m}_l \equiv \mathrm{m}_i a_{il} \ (\mathrm{mod}\ \mathrm{m}_i \mathrm{m}_l), \quad 1 \leq i < l \leq n.$$

Proof. By Lemma 4.9, $\mathcal{B}(V)'^{J_{\underline{a}}}$ is isomorphic to a Nichols algebra in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$ if and only if $g_i^{\mathrm{m}_i} \rhd'_{J_{\underline{a}}} X_j = X_j$ for all $1 \leq i \leq n, 1 \leq j \leq m$. By definition, we have

$$g_i^{\mathrm{m}_i} \rhd'_{J_{\underline{a}}} X_j = \frac{J_{\underline{a}}(g_i^{\mathrm{m}_i}, h_j)}{J_{\underline{a}}(h_j, g_i^{\mathrm{m}_i})} g_i^{\mathrm{m}_i} \rhd' X_j = \frac{J_{\underline{a}}(g_i^{\mathrm{m}_i}, h_j)}{J_{\underline{a}}(h_j, g_i^{\mathrm{m}_i})} \zeta_{m_i}^{\mathrm{m}_i x_{ij}} X_j$$

for all $1 \leq i \leq n, 1 \leq j \leq m$. Using (3.11),

$$\frac{J_{\underline{a}}(g_i^{\mathrm{m}_i}, h_j)}{J_{\underline{a}}(h_j, g_i^{\mathrm{m}_i})} \zeta_{m_i}^{\mathrm{m}_i x_{ij}} = \frac{1}{\zeta_{m_i}^{a_i s_{ji} \mathrm{m}_i} \prod_{i<k\leq n} \zeta_{\mathrm{m}_i \mathrm{m}_k}^{a_{ik} s_{ik} \mathrm{m}_i}} \zeta_{m_i}^{\mathrm{m}_i x_{ij}}.$$

So for all $1 \leq i \leq n, 1 \leq j \leq m$, equations $g_i^{\mathrm{m}_i} \triangleright'_{J_{\underline{a}}} X_j = X_j$ are equivalent to

$$(4.8) \qquad \zeta_{m_i}^{\mathrm{m}_i x_{ij}} = \zeta_{m_i}^{a_i s_{ji} \mathrm{m}_i} \prod_{i<k\leq n} \zeta_{\mathrm{m}_i \mathrm{m}_k}^{a_{ik} s_{jk} \mathrm{m}_i}.$$

Next we will show that equations (4.8) are equal to equations (4.5)–(4.7). At first, we assume that (4.8) hold. Then for any $1 \leq l \leq n$, we have

$$(4.9) \qquad \zeta_{m_i}^{\mathrm{m}_i x_{ij} t_{lj}} = \zeta_{m_i}^{a_i t_{lj} s_{ji} \mathrm{m}_i} \prod_{i<k\leq n} \zeta_{\mathrm{m}_i \mathrm{m}_k}^{a_{ik} t_{lj} s_{jk} \mathrm{m}_i}.$$

Considering the product on both sides of equations (4.9) for $j = 1, \ldots, m$, we get

$$\prod_{j=1}^m \zeta_{m_i}^{\mathrm{m}_i x_{ij} t_{lj}} = \prod_{j=1}^m \left[\zeta_{m_i}^{a_i t_{lj} s_{ji} \mathrm{m}_i} \prod_{i<k\leq n} \zeta_{\mathrm{m}_i \mathrm{m}_k}^{a_{ik} t_{lj} s_{jk} \mathrm{m}_i} \right],$$

which is equal to

$$(4.10) \qquad \zeta_{m_i}^{\sum_{j=1}^m \mathrm{m}_i x_{ij} t_{lj}} = \zeta_{m_i}^{a_i \sum_{j=1}^m t_{lj} s_{ji} \mathrm{m}_i} \prod_{i<k\leq n} \zeta_{\mathrm{m}_i \mathrm{m}_k}^{a_{ik} \sum_{j=1}^m t_{lj} s_{jk} \mathrm{m}_i}.$$

When $i > l$, by (4.4), equations (4.10) become

$$(4.11) \qquad \zeta_{m_i}^{\sum_{j=1}^m \mathrm{m}_i x_{ij} t_{lj}} = 1.$$

These imply that $\sum_{j=1}^m x_{ij} t_{lj} \equiv 0 \pmod{\mathrm{m}_i}$, which are equations (4.5). When $i = l$, then equations (4.10) become

$$(4.12) \qquad \zeta_{m_i}^{\sum_{j=1}^m \mathrm{m}_i x_{ij} t_{ij}} = \zeta_{m_i}^{a_i \mathrm{m}_i},$$

which imply equations (4.6). When $i < l$, then equations (4.10) become

$$(4.13) \qquad \zeta_{m_i}^{\sum_{j=1}^m \mathrm{m}_i x_{ij} t_{lj}} = \zeta_{\mathrm{m}_i \mathrm{m}_l}^{a_{il} \mathrm{m}_i},$$

which are the same as equations (4.7).

Next, we assume that (4.5)–(4.7) hold. Clearly, equations (4.5)–(4.7) are equal to (4.11)–(4.13). Considering the product of these three identities, we get

$$(4.14) \qquad \prod_{j=1}^m \zeta_{m_i}^{\mathrm{m}_i x_{ij} t_{lj}} = \prod_{j=1}^m \left[\zeta_{m_i}^{a_i t_{lj} s_{ji} \mathrm{m}_i} \prod_{i<k\leq n} \zeta_{\mathrm{m}_i \mathrm{m}_k}^{a_{ik} t_{lj} s_{jk} \mathrm{m}_i} \right]$$

for $1 \leq i, l \leq n$. Let $\mathbf{T}'$ be an $m \times m$-matrix such that $t'_{ij} = t_{ij}$ for $1 \leq i \leq n$, $1 \leq j \leq m$ and otherwise $t'_{ij} = 0$. From (4.4) we know that the rank of $\mathbf{T}'$ is n. So for any $1 \leq i \leq n$,

there exists an $m \times m$-matrix $\mathbf{S}_i = (s^i_{jk})$ such that $\sum_{k=1}^{m} s^i_{jk} t'_{kl} = \delta_{ij}\delta_{il}$ for all $1 \leq j, l \leq m$. Taking s^r_{rl}'s power of equation (4.14), we have

$$\prod_{j=1}^{m} \zeta_{m_i}^{m_i x_{ij} s^r_{rl} t_{lj}} = \prod_{j=1}^{m} [\zeta_{m_i}^{a_i s^r_{rl} t_{lj} s_{ji} m_i} \prod_{i<k\leq n} \zeta_{m_i m_k}^{a_{ik} s^r_{rl} t_{lj} s_{jk} m_i}].$$

By taking the product of the identity above for $1 \leq l \leq m$, we get

$$(4.15) \qquad \prod_{l=1}^{m} \prod_{j=1}^{m} \zeta_{m_i}^{m_i x_{ij} s^r_{rl} t_{lj}} = \prod_{l=1}^{m} \prod_{j=1}^{m} [\zeta_{m_i}^{a_i s^r_{rl} t_{lj} s_{ji} m_i} \prod_{i<k\leq n} \zeta_{m_i m_k}^{a_{ik} s^r_{rl} t_{lj} s_{jk} m_i}].$$

The left-hand side of the identity is

$$\prod_{l=1}^{m} \prod_{j=1}^{m} \zeta_{m_i}^{m_i x_{ij} s^r_{rl} t_{lj}} = \prod_{l=1}^{m} \prod_{j=1}^{m} \zeta_{m_i}^{m_i x_{ij} s^r_{rl} t'_{lj}} = \zeta_{m_i}^{m_i x_{ir}}.$$

The right-hand side is

$$\prod_{l=1}^{m} \prod_{j=1}^{m} [\zeta_{m_i}^{a_i s^r_{rl} t_{lj} s_{ji} m_i} \prod_{i<k\leq n} \zeta_{m_i m_k}^{a_{ik} s^r_{rl} t_{lj} s_{jk} m_i}]$$

$$= \prod_{l=1}^{m} \prod_{j=1}^{m} [\zeta_{m_i}^{a_i s^r_{rl} t'_{lj} s_{ji} m_i} \prod_{i<k\leq n} \zeta_{m_i m_k}^{a_{ik} s^r_{rl} t'_{lj} s_{jk} m_i}]$$

$$= \zeta_{m_i}^{a_i s_{ri} m_i} \prod_{i<k\leq n} \zeta_{m_i m_k}^{a_{ik} s_{rk} m_i}.$$

Hence (4.15) is actually identical to (4.8). □

Remark 4.12. As we pointed out at the paragraph after Corollary 3.13, we can assume that $m_i | m_j$ for $i < j$. *We will keep this assumption in the rest of the paper.* In this way, the identities in (4.7) are equal to

$$\frac{m_l}{m_i} \sum_{j=1}^{m} x_{ij} t_{lj} \equiv a_{il} \pmod{m_l}, \quad 1 \leq i < l \leq n.$$

The above proposition implies that we do not have many choices on the sequence $\underline{a} \in A$, see (3.9).

Corollary 4.13. *For the Nichols algebra $\mathcal{B}(V)' \in {}^G_G \mathcal{YD}$, there is at most one $\underline{a} \in A$ such that $\mathcal{B}(V)'^{J_{\underline{a}}}$ is isomorphic to a Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}} \mathcal{YD}^{\Phi_{\underline{a}}}$ through π. Moreover, this $\underline{a}$ exists if and only if equations (4.5) hold and in this case $\underline{a}$ can be taken in the following way:*

$$(4.16) \qquad a_i \equiv \sum_{j=1}^{m} x_{ij} t_{ij} \pmod{m_i}; \quad a_{il} \equiv \frac{m_l}{m_i} \sum_{j=1}^{m} x_{ij} t_{lj} \pmod{m_l}; \quad a_{ilt} = 0$$

for $1 \leq i \leq n$, $1 \leq i < l \leq n$ and $1 \leq i < l < t \leq n$.

Proof. By Proposition 4.11, we know that $\underline{a}$ must satisfy equations (4.6) and (4.7). At the same time, since we always assume that $\Phi_{\underline{a}}$ is an abelian cocycle, $a_{rst} = 0$ for all $1 \leq r < s < t \leq n$ by Proposition 3.14. Therefore, there is at most one $\underline{a}$ that satisfies these conditions. Proposition 4.11 also implies that such an $\underline{a}$ exists if and only if equations (4.5) hold. $\qquad\square$

Now we are in the position to find the "return trip" as follows:

$$(4.17) \qquad\qquad \mathcal{B}(V) \in {}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$$

$$\Big\uparrow \text{(4.5) hold}$$

$$\mathcal{B}(V)'^{J_{\underline{a}}} \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$$

$$\Big\uparrow \text{by (4.16), find } J_{\underline{a}}$$

$$\text{usual Nichols algebra } \mathcal{B}(V)' \in {}^{G}_{G}\mathcal{YD}$$

4.4. Root datum and classification of Nichols algebras of diagonal type with arithmetic root systems in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$.
By (4.17), we want to formulate the conditions listed in (4.17) by the language of root data. Using such language, we get a complete classification of Nichols algebras of diagonal type with arithmetic root systems in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$.

Now suppose $(\triangle, \chi, E)$ is an arithmetic root system, and $\mathcal{D}_{\chi,E}$ is the Dynkin diagram of $(\triangle, \chi, E)$. Up to twist equivalence, $(\triangle, \chi, E)$ is uniquely determined by $\mathcal{D}_{\chi,E}$. In [20], Heckenberger classified all the arithmetic root systems. Fix a Dynkin diagram with m vertices. We call

$$\{q_{ii} = \chi(e_i, e_i), \ \widetilde{q_{ij}} = \chi(e_i, e_j)\chi(e_j, e_i) \mid 1 \leq i, j \leq m\}$$

the structure constants of $\mathcal{D}_{\chi,E}$.

Definition 4.14. Let $\mathbb{G} = \mathbb{Z}_{\mathrm{m}_1} \times \cdots \times \mathbb{Z}_{\mathrm{m}_n}$ be the abelian group defined as above and set $m_i := \mathrm{m}_i^2$ for $1 \leq i \leq n$. Suppose $\mathcal{D}_{\chi,E}$ is a Dynkin diagram of an arithmetic root system $\triangle_{\chi,E}$ and $(q_{ii}, \widetilde{q_{ij}})$ is the set of structure constants. Moreover, suppose there exist parameter matrices $\mathbf{S}$ and $\mathbf{X}$ such that

(1) $\mathbf{S} = (s_{ij})_{m \times n}$ is a matrix with integer entries $0 \leq s_{ij} < m_i$ for all $1 \leq i \leq m, 1 \leq j \leq n$ such that there exists a matrix $\mathbf{T} = (t_{ij})_{n \times m}$ satisfying (4.4);

(2) $\mathbf{X} = (x_{ij})_{n \times m}$ is a matrix with integer entries $0 \leq x_{ij} < m_i$ for all $1 \leq i \leq m, 1 \leq j \leq n$ such that

$$q_{ii} = \prod_{k=1}^{n} \zeta_{m_k}^{s_{ik}x_{ki}}, \quad \widetilde{q_{ij}} = \prod_{k=1}^{n} \zeta_{m_k}^{s_{ik}x_{kj}+s_{jk}x_{ki}}$$

and satisfying (4.5).

Then we call $\mathfrak{D} = \mathfrak{D}(\mathcal{D}_{\chi,E}, \mathbf{S}, \mathbf{X})$ a root datum over $\mathbb{G}$ and moreover we call $\triangle$ (resp. $\triangle_{\chi,E}$) the root system (resp. arithmetic root system) of $\mathfrak{D}$.

For a fixed root datum $\mathfrak{D} = \mathfrak{D}(\mathcal{D}_{\chi,E}, \mathbf{S}, \mathbf{X})$ over $\mathbb{G}$, define a sequence $\underline{a} \in A$ through equations (4.16). Now we can define a Nichols algebra $\mathcal{B}(V_{\mathfrak{D}}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\bar{\Phi}_{\underline{a}})}$ in the fol-

lowing way: Let $V_{\mathfrak{D}}$ be the Yetter–Drinfeld module in ${}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$ with a canonical basis $\{X_i \mid 1 \leq i \leq m\}$ such that

$$\delta_L(X_i) = \prod_{k=1}^{n} g_k^{s_{ik}} \otimes X_i, \quad g_i \blacktriangleright X_j = \zeta_{m_i}^{x_{ij}} \frac{J_{\underline{a}}(g_i, \prod_{k=1}^{n} g_k^{s_{ik}})}{J_{\underline{a}}(\prod_{k=1}^{n} g_k^{s_{ik}}, g_i)} X_j.$$

Now the main result of this section can be stated as follows.

Theorem 4.15.   (1) *The Nichols algebra $\mathcal{B}(V_{\mathfrak{D}})$ is isomorphic to a Nichols algebra of diagonal type with arithmetic root system in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$ through the group epimorphism $\pi : G \to \mathbb{G}$.*

(2) *Suppose $\mathcal{B}(V)$ is a Nichols algebra of diagonal type with arithmetic root system in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$ and the support group is $\mathbb{G}$. Then there exists a root datum $\mathfrak{D}$ over $\mathbb{G}$ such that $\mathcal{B}(V_{\mathfrak{D}}) \cong \mathcal{B}(V)$ through the group epimorphism $\pi : G \to \mathbb{G}$.*

Proof.   The first statement is just a direct consequence of (4.17) and the definition of a root datum. Now we show the second statement. According to (4.3), from $\mathcal{B}(V)$ one can construct a usual Nichols algebra $\mathcal{B}(V)'$. By the construction of $\mathcal{B}(V)'$ we can find that $\mathcal{B}(V)'^{J_{\underline{a}}}$ is isomorphic to $\mathcal{B}(V)$ through π. By Proposition 4.11 and the definition of a root datum, we know that there is a root datum $\mathfrak{D}$ over $\mathbb{G}$ such that $\mathcal{B}(V)'^{J_{\underline{a}}} = \mathcal{B}(V_{\mathfrak{D}})$. □

Convention 4.16.   By this theorem, we know that the Nichols algebra $\mathcal{B}(V_{\mathfrak{D}})$ is isomorphic to a unique Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$. For convenience, this Nichols algebra is denoted by $\mathcal{B}(\mathfrak{D})$.

5. Classification results

In this section, all the finite-dimensional connected graded pointed Majid algebras $\mathbb{M}$ of diagonal type will be classified. The main idea is to show that the coinvariant subalgebra of $\mathbb{M}$ is indeed a Nichols algebra of diagonal type and from this we can apply the classification results obtained in the previous section.

5.1. General setup.   In this section, we always assume that $\mathbb{M}$ is a finite-dimensional connected coradically graded pointed Majid algebra of diagonal type. From Section 2.1, we know that

$$\mathbb{M}_0 = (\mathbf{k}\mathbb{G}, \Phi)$$

where $\mathbb{G}$ is the group consisting of all the group-like elements and Φ is a 3-cocycle on $\mathbb{G}$. Using the same arguments given in Proposition 4.1 and our assumption that $\mathbb{M}$ being connected, we know that $\mathbb{G}$ is abelian and Φ is an abelian cocycle. Therefore,

$$\mathbb{G} = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle$$

with $m_i | m_j$ for $1 \leq i < j \leq n$, and

$$\Phi = \Phi_{\underline{a}}$$

for some $\underline{a} \in A$ with $a_{rst} = 0$ for all $1 \leq r < s < t \leq n$.

Let $\mathbb{R}$ be the coinvariant subalgebra of $\mathbb{M}$. Then $\mathbb{R}$ is a Hopf algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$ and

$$\mathbb{M} = \mathbb{R}\#\mathbf{k}\mathbb{G}.$$

The main task of this section is to show that $\mathbb{R}$ is indeed a Nichols algebra in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi}$. From the classification results obtained in the previous section, we can classify $\mathbb{M}$ directly.

5.2. $\mathbb{R}$ is a Nichols algebra. Note that we already showed that each finite-dimensional rank-2 pointed Majid algebra is generated by group-like and skew-primitive elements in [23]. We gradually realize that the methods developed in [23] still work for pointed Majid algebras of diagonal type. For completeness and safety, the proof will be given though it is similar to the version of [23].

The main result of this subsection can be stated as follows.

Proposition 5.1. *In ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$, we have $\mathbb{R} \cong \mathcal{B}(\mathbb{R}_1)$.*

We give several preparations. Take a Nichols algebra $\mathcal{B}(V)$ of diagonal type in ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$. Then according to (4.3), we have

$$\mathcal{B}(V) \cong \mathcal{B}(\widetilde{V}) \in {}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$$

and $\mathcal{B}(\widetilde{V})^{J_{\underline{a}}^{-1}} = \mathcal{B}(\widetilde{V}^{J_{\underline{a}}^{-1}})$ is a usual Nichols algebra in ${}^{G}_{G}\mathcal{YD}$. As before, we denote this usual Nichols algebra by $\mathcal{B}(V)'$ and we use V' to denote the Yetter–Drinfeld module $\widetilde{V}^{J_{\underline{a}}^{-1}}$. That is, $\mathcal{B}(V)' = \mathcal{B}(V')$. Let $\{X_i \mid 1 \leq i \leq m\}$ be a canonical basis of V'. Then $\mathcal{B}(V') \cong T(V')/I$ where I is the Hopf ideal of $T(V')$ generated by the polynomials in $\{X_i \mid 1 \leq i \leq m\}$ listed in [5, Theorem 3.1]. In the following, let $\mathbf{S}$ denote the set of these polynomials. Define a map $\Psi : T_{\partial(J_{\underline{a}})}(\widetilde{V}) = T_{\partial(J_{\underline{a}})}(V'^{J_{\underline{a}}}) \to T(V')$ by

$$\Psi\big((\cdots((Y_1 \circ Y_2) \circ Y_3)\cdots Y_n)\big) = \prod_{i=1}^{n-1} J_{\underline{a}}(y_1 \cdots y_i, y_{i+1})Y_1 Y_2 \cdots Y_n$$

for all $Y_i \in \{X_1, X_2, \ldots, X_m\}$. It is easy to see that Ψ is an isomorphism of linear spaces. The following conclusion is [23, Lemma 4.5].

Lemma 5.2. *The set $\Psi^{-1}(\mathbf{S})$ is a minimal set of defining relations of $\mathcal{B}(V')^{J_{\underline{a}}} = \mathcal{B}(\widetilde{V})$.*

We also need the following two lemmas, which were given essentially in [5] and were rephrased as follows in [23, Lemmas 4.6 and 4.7].

Lemma 5.3. *Let Z be a polynomial in $\mathbf{S}$. Then*

$$\mathcal{B}(V'^{J_{\underline{a}}} \oplus \mathbf{k}\Psi^{-1}(Z)) \cong \mathcal{B}(V' \oplus \mathbf{k}Z)^{J_{\underline{a}}}.$$

Lemma 5.4. *Let $\mathcal{B}(V') \in {}^{G}_{G}\mathcal{YD}$ be a finite-dimensional Nichols algebra of diagonal type, Z a polynomial in $\mathbf{S}$ and $U' = V' \oplus \mathbf{k}Z$. Then $\mathcal{B}(U')$ is infinite-dimensional.*

The following is a generalized version of [23, Propostion 4.8], where we proved it in the rank-2 case.

Proposition 5.5. *Let $R = \bigoplus_{i \geq 0} R_i$ be a finite-dimensional graded (not necessarily coradically graded) Hopf algebra in $_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$. If R is generated by R_1, then $R = \mathcal{B}(R_1)$.*

Proof. Let I be an ideal of $T_{\Phi}(R_1)$ such that $R = T_{\Phi}(R_1)/I$. Clearly, we have a surjective Hopf map

$$\theta : R \twoheadrightarrow \mathcal{B}(R_1).$$

By Proposition 4.7, $\mathcal{B}(R_1)$ is also a Nichols algebra in $_G^G\mathcal{YD}^{\pi^*(\Phi)}$ for $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$. By Proposition 3.15, $\pi^*(\Phi) = \partial(J)$. Therefore, $\mathcal{B}(R_1)^{J^{-1}} \in {}_G^G\mathcal{YD}$ is a usual Nichols algebra. Now assume that θ is not an isomorphism. Then there should be some polynomials in $\Psi^{-1}(\mathbf{S})$, which are not contained in I by Lemma 5.2. Suppose that $\Psi^{-1}(Z)$ is one of those with minimal length. Then we know that $\Psi^{-1}(Z)$ must be a primitive element in R. Let $U = R_1 \oplus \mathbf{k}\Psi^{-1}(Z)$. Then, by the preceding assumption, there is an embedding of linear spaces $\mathcal{B}(U) \hookrightarrow R$.

We already know that $\mathcal{B}(R_1)^{J^{-1}}$ is a finite-dimensional Nichols algebra in $_G^G\mathcal{YD}$. It is not hard to see that there exists $R_1' \in {}_G^G\mathcal{YD}$ such that $R_1 = R_1'^J$ (since J induces an equivalence between $_G^G\mathcal{YD}$ and $_G^G\mathcal{YD}^{\partial(J)} = {}_G^G\mathcal{YD}^{\pi^*(\Phi)}$). By Lemma 5.3, we have

$$\mathcal{B}(R_1' \oplus Z)^J = \mathcal{B}(R_1 \oplus \mathbf{k}\Psi^{-1}(Z)) = \mathcal{B}(U).$$

Note that $\mathcal{B}(R_1' \oplus Z)^J$ is infinite-dimensional due to Lemma 5.4. Hence $\mathcal{B}(U)$ is infinite-dimensional, which contradicts the assumption that R is finite-dimensional. Thus θ is an isomorphism and R is the Nichols algebra $\mathcal{B}(R_1)$. $\qquad\square$

In order to prove Proposition 5.1, we still need the following lemma.

Lemma 5.6. *Let $R = \bigoplus_{i \geq 0} R_i$ be a graded Hopf algebra in $_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi}$ with $R_0 = \mathbf{k}1$ and $P(R) = R_1$. Then the right dual $R^* = \bigoplus_{i \geq 0} R_i^*$ (resp. the left dual $^*R = \bigoplus_{i \geq 0} {}^*R_i$) is generated by R_1^* (resp. $^*R_1$).*

Proof. Note that the proof of [23, Lemma 4.10] does neither depend on the abelian group G nor on the abelian 3-cocycle Φ of G. Hence we can prove the lemma in the same way. $\qquad\square$

Proof of Proposition 5.1. By assumption, $\mathbb{R}_0 = \mathbf{k}1$ and $P(\mathbb{R}) = \mathbb{R}_1$. According to Lemma 5.6, $\mathbb{R}^* = \bigoplus_{i \geq 0} \mathbb{R}_i^*$ is generated by $\mathbb{R}_1^*$. By Proposition 5.5, $\mathbb{R}^* = \mathcal{B}(\mathbb{R}_1^*)$. So we have $P(\mathbb{R}^*) = \mathbb{R}_1^*$, and $^*(\mathbb{R}^*) = \mathbb{R}$ is generated by $\mathbb{R}_1$ according to Lemma 5.6 again. Hence $\mathbb{R}$ is also a Nichols algebra by Proposition 5.5. Thus, $\mathbb{R} = \mathcal{B}(\mathbb{R}_1)$. $\qquad\square$

5.3. Classification result. For a root datum $\mathfrak{D} = \mathfrak{D}(\mathcal{D}_{\chi,E}, \mathbf{S}, \mathbf{X})$, there is a Nichols algebra $\mathcal{B}(\mathfrak{D})$ in $_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$, where $\underline{a} \in A$ is determined by equations (4.16). If we denote $\mathbb{M}(\mathfrak{D}) = \mathcal{B}(\mathfrak{D})\#\mathbf{k}\mathbb{G}$, we can formulate the main result of the paper as follows.

Theorem 5.7. *Keep the notations as before. We have:*

(1) *The Majid algebra $\mathbb{M}(\mathfrak{D})$ is a connected coradical graded pointed Majid algebra of diagonal type over the group $\mathbb{G}$. Moreover, $\mathbb{M}(\mathfrak{D})$ is finite-dimensional if and only if the heights of all restricted Poincaré–Birkhoff–Witt generators of $\mathcal{B}(\mathfrak{D})$ are finite.*

(2) *Any finite-dimensional connected coradical graded pointed Majid algebra of diagonal type over $\mathbb{G}$ is isomorphic to a $\mathbb{M}(\mathfrak{D})$ for some $\mathfrak{D}$.*

Proof.   It follows from Proposition 5.1 and Theorem 4.15.   □

5.4. A corollary.   In [2, Conjecture 1.4], Andruskiewitch–Schneider conjectured that every finite-dimensional pointed Hopf algebras over **k** is generated by group-like and skew-primitive elements. This is the so-called generation problem, which plays an important role in the classification of pointed Hopf algebras. It is true in many cases, see [5]. This conjecture was generalized to finite-dimensional pointed Majid algebras or even to pointed finite tensor categories [14].

Corollary 5.8.   *Suppose $\mathbb{M}$ is a finite-dimensional pointed Majid algebras of diagonal type. Then $\mathbb{M}$ is generated by group-like and skew-primitive elements.*

Proof.   Since $\mathbb{M}$ is generated by group-like and skew-primitive elements if and only if its coradically graded version $\mathrm{gr}(\mathbb{M})$ is, we can assume that $\mathbb{M}$ is coradically graded. Let $\mathbb{R}$ be the coinvariant subalgebra of $\mathbb{M}$, and assume that its support group is H. Then $\mathbb{R}\#\mathbf{k}H$ is a finite-dimensional connected coradically graded pointed Majid algebra of diagonal type, and thus it is generated by group-like and skew-primitive elements according to Theorem 5.7. This implies that $\mathbb{M}$ is also generated by group-like and skew-primitive elements.   □

6. Examples of genuine pointed Majid algebras

In this section, we provide some methods to construct genuine pointed Majid algebras from arithmetic root systems. For each arithmetic root system $\bigtriangleup_{\chi,E}$ of rank θ satisfying a mild condition, we show that there always exists a genuine pointed Majid algebra of standard type $\mathbb{M} \cong \mathcal{B}(V)\#\mathbf{k}\mathbb{Z}_m^\theta$, such that the arithmetic root system of $\mathcal{B}(V)$ is $\bigtriangleup_{\chi,E}$. For arithmetic root systems of Cartan type, we also provide a unified method to construct genuine finite-dimensional pointed Majid algebras.

6.1. Pointed Majid algebras of typical type.   Suppose that $\mathbb{M}$ is a pointed Majid algebra generated by the abelian group $\mathbb{G}$ and skew-primitive elements $\{X_1, \ldots, X_n\}$ with $\Delta(X_i) = X_i \otimes 1 + \mathrm{g}_i \otimes X_i$, $1 \leq i \leq n$. Then as in our previous paper [23], we say that $\mathbb{M}$ is of *typical* type if $\mathbb{G} = \langle \mathrm{g}_1 \rangle \times \cdots \times \langle \mathrm{g}_n \rangle$. This definition is transferred naturally to Yetter–Drinfeld modules and thus Nichols algebras.

Definition 6.1.   A Yetter–Drinfeld module V in $_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$ of diagonal type is said to be of *typical* type if there exists a canonical basis $X_1, \ldots, X_n$ with degrees $\mathrm{g}_1, \ldots, \mathrm{g}_n$, respectively, such that $\mathbb{G} = \langle \mathrm{g}_1 \rangle \times \cdots \times \langle \mathrm{g}_n \rangle$. The Nichols algebra $\mathcal{B}(V) \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$ is typical if V is so.

The definition is independent of the choice of the canonical basis. If $V \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$ is a Yetter–Drinfeld module of typical type, then $\mathcal{B}(V)\#\mathbf{k}\mathbb{G}$ is a pointed Majid algebra of typical type.

Lemma 6.2. *Let* $\mathbb{G} = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ *and let* $\Phi_{\underline{a}}$ *be an abelian 3-cocycle on* $\mathbb{G}$.

(1) *Suppose* V *is a Yetter–Drinfeld module of typical type in* ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$ *and* $X_1, \ldots, X_n$ *is a canonical basis of* V. *Let* (x_{ij}) *be the numbers satisfying* $g_i \triangleright X_j = \zeta_{m_i^2}^{x_{ij}} X_j$ *for all* $1 \leq i, j \leq n$. *Then we have*

(6.1) $\quad x_{ij} \equiv 0 \ (\mathrm{mod} \ m_i), \quad i > j,$

(6.2) $\quad x_{ii} \equiv a_i \ (\mathrm{mod} \ m_i); \quad m_j x_{ij} \equiv m_i a_{ij} \ (\mathrm{mod} \ m_i m_j), \quad i < j.$

(2) *Conversely, let* $V = \mathbf{k}\{X_1, \ldots, X_n\}$ *be a* $\mathbf{k}\mathbb{G}$*-comodule such that the degree of* X_i *is* g_i *for* $1 \leq i \leq n$. *If we have numbers* $(x_{ij})_{1 \leq i, j \leq n}$ *satisfying equations* (6.1), *then the action*

$$g_i \triangleright X_j := \zeta_{m_i^2}^{x_{ij}} X_j, \quad 1 \leq i, j \leq n$$

makes V *a typical Yetter–Drinfeld module in* ${}^{\mathbb{G}}_{\mathbb{G}}\mathcal{YD}^{\Phi_{\underline{a}}}$, *where the sequence* $\underline{a}$ *are the numbers determined by equations* (6.2).

Proof. (1) Let $G = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle$ be such that $|g_i| = m_i^2$, $1 \leq i \leq n$. By Proposition 4.7, $\mathcal{B}(V)$ can be viewed as a Yetter–Drinfeld module in ${}^{G}_{G}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$ through

$$g_i \triangleright X_j = \zeta_{m_i^2}^{x_{ij}} X_j, \quad 1 \leq i, j \leq n.$$

So $\mathcal{B}(V)^{J_{\underline{a}}^{-1}} \in {}^{G}_{G}\mathcal{YD}$. Since

$$g_i \triangleright_{J_{\underline{a}}^{-1}} X_j = \frac{J(g_i, g_j)}{J(g_j, g_i)} g_i \triangleright X$$

and $J(g_i, g_j) = J(g_j, g_i) = 1$, we have

$$g_i \triangleright_{J_{\underline{a}}^{-1}} X_j = \zeta_{m_i^2}^{x_{ij}} X_j.$$

Identities (6.1) and (6.2) follow from Proposition 4.11.

(2) Follows from Proposition 4.11. $\qquad \square$

In the following, for a root of unity q we use $|q|$ to denote its order.

Proposition 6.3. *Let* $\triangle_{\chi, E}$ *be a connected arithmetic root system of rank* θ *listed in* [20] *and let* $\{q_{ii}, \widetilde{q_{ij}} \mid 1 \leq i < j \leq \theta\}$ *be its structure constants.*

(a) *If there is a* q_{ii} *or* $\widetilde{q_{ij}}$ *such that its order is not of the form* $p_1 p_2 \cdots p_n$, *where* $p_1, \ldots, p_n$ *are mutually distinct prime numbers, then there is a typical Yetter–Drinfeld module*

$$V \in {}^{\mathbf{k}\mathbb{Z}_m^\theta}_{\mathbf{k}\mathbb{Z}_m^\theta}\mathcal{YD}^{\Phi_{\underline{a}}}$$

for some m *and* $\Phi_{\underline{a}}$ *such that* $\mathcal{B}(V) \# \mathbf{k}\mathbb{Z}_m^\theta$ *is a genuine pointed Majid algebra of typical type, and the arithmetic root system of* $\mathcal{B}(V)$ *is equal to* $\triangle_{\chi, E}$.

(b) *If the order of each q_{ii} and $\widetilde{q_{ij}}$ is of the form $p_1 p_2 \cdots p_n$ for some distinct prime numbers $p_1, p_1, \ldots, p_n$, then there is no typical Yetter–Drinfeld module in*

$$\mathbf{k}\mathbb{Z}_m^\theta \mathcal{YD}^{\Phi_{\underline{a}}}_{\mathbf{k}\mathbb{Z}_m^\theta}$$

for all $m \geq 1$ and $\Phi_{\underline{a}} \neq 1$.

Proof. (a) Firstly, we define a function $\upsilon : \mathbb{N} \to \mathbb{N}$ by

$$\upsilon(k) = \begin{cases} k+1 & \text{if } k \text{ is odd,} \\ k & \text{if } k \text{ is even.} \end{cases}$$

Let Υ be the map

$$\Upsilon : \mathbb{N} \to \mathbb{N}, \quad k = p_1^{N_1} p_2^{N_2} \cdots p_n^{N_n} \mapsto \Upsilon(k) = p_1^{\upsilon(N_1)} p_2^{\upsilon(N_2)} \cdots p_n^{\upsilon(N_n)}.$$

Here $p_1, p_2, \ldots, p_n$ are mutually distinct prime numbers. Let $m_i = |q_{ii}|$, $m_{ij} = |\widetilde{q_{ij}}|$ for $1 \leq i, j \leq \theta$. By m we denote the minimal positive number such that $\Upsilon(m_i)|\mathrm{m}$ and $\Upsilon(m_{ij})|\mathrm{m}$ for all $1 \leq i, j \leq \theta$. Then it is obvious that $\sqrt{\mathrm{m}}$ is a positive integer, since all $\sqrt{\Upsilon(m_i)}$ and $\sqrt{\Upsilon(m_{ij})}$, $1 \leq i, j \leq \theta$ are integers. Let $m = \sqrt{\mathrm{m}}$.

Next we will show that there is a typical Yetter–Drinfeld module

$$V \in \mathbf{k}\mathbb{Z}_m^\theta \mathcal{YD}^{\Phi_{\underline{a}}}_{\mathbf{k}\mathbb{Z}_m^\theta}$$

for some nontrivial 3-cocycle $\Phi_{\underline{a}}$ on $\mathbb{Z}_m^\theta$. Suppose $\mathrm{g}_1, \ldots, \mathrm{g}_\theta$ are free generators of $\mathbb{Z}_m^\theta$, that is, $\mathbb{Z}_m^\theta = \langle \mathrm{g}_1 \rangle \times \cdots \times \langle \mathrm{g}_\theta \rangle$, and $V = \mathbf{k}\{X_1, \ldots, X_\theta\}$ is a $\mathbb{Z}_m^\theta$-graded vector space. For $1 \leq i, j \leq \theta$, define

$$x_{ij} = \begin{cases} \frac{m^2}{m_i} & \text{if } i = j, \\ 0 & \text{if } \widetilde{q_{ij}} = 1, i \neq j, \\ 0 & \text{if } \widetilde{q_{ij}} \neq 1 \text{ and } i > j, \\ \frac{m^2}{m_{ij}} & \text{if } \widetilde{q_{ij}} \neq 1 \text{ and } i < j. \end{cases}$$

According to (2) of Lemma 6.2, $\mathrm{g}_i \triangleright X_j = \zeta_{m^2}^{x_{ij}} X_j$, $1 \leq i, j \leq \theta$ makes V a typical Yetter–Drinfeld module in

$$\mathbf{k}\mathbb{Z}_m^\theta \mathcal{YD}^{\Phi_{\underline{a}}}_{\mathbf{k}\mathbb{Z}_m^\theta},$$

where the sequence $\underline{a}$ is determined by equations (6.2). Since

$$\zeta_{m^2}^{x_{ii}} = \zeta_{m^2}^{\frac{m^2}{m_i}} = q_{ii}, \quad \zeta_{m^2}^{x_{ij}} \zeta_{m^2}^{x_{ji}} = \zeta_{m^2}^{\frac{m^2}{m_{ij}}} = \widetilde{q_{ij}}$$

for $1 \leq i, j \leq \theta$, we prove that the arithmetic root system of $\mathcal{B}(V)$ is equal to $\triangle_{\chi, E}$.

At last, we will show that $\underline{a}$ is nonzero. From the assumption of the first part of the proposition, there is an element ζ in $\{q_{ii}, \widetilde{q_{ij}}\}$ satisfying the following conditions:

(C1) $|\zeta| = p_1^{N_1} p_2^{N_2} \cdots p_n^{N_n}$ and there exists some l such that $N_l \geq 2$. Here $p_1, p_2, \ldots, p_n$ are mutually distinct prime numbers.

(C2) $p_l^{N_l + 1} \nmid m_i$, $p_l^{N_l + 1} \nmid m_{ij}$ if $m_i \neq |\zeta|$, $m_{ij} \neq |\zeta|$.

If $\zeta = q_{ii}$ for some i, then by the definition of m and the choice of ζ, we have $m \nmid \frac{m^2}{m_i}$, which implies $a_i \not\equiv 0 \pmod{m}$ by equations (6.2). Similarly, if $\zeta = \widetilde{q_{ij}}$ for some i, j, then one can prove that $a_{ij} \not\equiv 0 \pmod{m}$ by (6.2). We have proved that $\underline{a}$ is nonzero.

(b) Suppose there is a typical Yetter–Drinfeld module

$$V \in {}^{\mathbf{k}\mathbb{Z}_m^\theta}_{\mathbf{k}\mathbb{Z}_m^\theta}\mathcal{YD}^{\Phi_{\underline{a}}}$$

for some $m \geq 1$ and $\Phi_{\underline{a}}$. Then we will prove that $\Phi_{\underline{a}} = 1$.

On the one hand, let $\{X_1, \ldots, X_\theta\}$ be a canonical basis of V and $\{g_1, \ldots, g_\theta\}$ the corresponding degrees. Since V is a typical Yetter–Drinfeld module, $\mathbb{Z}_m^\theta = \langle g_1 \rangle \times \cdots \times \langle g_\theta \rangle$. Let $(x_{ij})_{1 \leq i,j \leq \theta}$ be the numbers defined by $g_i \triangleright X_j = \zeta_{m^2}^{x_{ij}} X_j$, $0 \leq x_{ij} < m^2$. So by equations (6.2), we have

$$(6.3) \qquad x_{ii} \equiv a_i \pmod{m}, \quad 1 \leq i \leq \theta,$$

$$(6.4) \qquad x_{il} \equiv a_{il} \pmod{m}, \quad 1 \leq i < l \leq \theta.$$

On the other hand, since the order of $\zeta_{m^2}^{x_{ii}} = q_{ii}$ is of the form $p_1 \cdots p_n$, where $p_1, \ldots, p_n$ are mutually distinct prime numbers, we have $p_1 \cdots p_n | m$, and hence $m | x_{ii}$. This implies that $a_i = 0$ by equations (6.3). Similarly, from equations (6.4) one can show that $a_{ij} = 0$ for $1 \leq i < j \leq \theta$. $\qquad \square$

According to Proposition 6.3, we can construct a big class of genuine pointed Majid algebras such that the corresponding Nichols algebras have arithmetic root systems.

Example 6.4. Let $\triangle_{\chi,E}$ be an arithmetic root system of the following type:

(1) rank-2 arithmetic root systems of cases 1–5, 7–12, 14 as listed in [20, Table 1],

(2) rank-3 arithmetic root systems of cases 1–8, 10, 18 as listed in [20, Table 2],

(3) rank-4 arithmetic root systems of cases 1–14, 22 as listed in [20, Table 3],

(4) higher rank (≥ 5) arithmetic root systems of cases 1–4, 7–10, 14, 19, 22 as listed in [20, Table 4],

such that the parameter q (if there is a parameter q in the root system) is a root of unity, and the order of q is of the form $p_1^{N_1} p_2^{N_2} \cdots p_n^{N_n}$, where $p_1, p_2, \ldots, p_n$ are mutually distinct prime numbers, $n \geq 3$ and there exists at least one $N_i \geq 2$ for some $1 \leq i \leq n$. Then we can construct a genuine pointed Majid algebra of typical type $\mathbb{M} = \mathcal{B}(V) \# \mathbf{k}\mathbb{Z}_m^\theta$ such that the arithmetic root system of $\mathcal{B}(V)$ is $\triangle_{\chi,E}$, where the number m is listed in Table 1.

Explicitly, let $\mathbb{Z}_m^\theta = \langle g_1 \rangle \times \cdots \times \langle g_\theta \rangle$ and let $V = \mathbf{k}\{X_1, \ldots, X_\theta\}$ be a $\mathbb{Z}_m^\theta$-graded vector space with $\deg X_i = g_i$ for $1 \leq i \leq \theta$. Define

$$g_i \triangleright X_j = \begin{cases} q_{ii} X_i & \text{if } i = j, \\ \widetilde{q_{ij}} X_j & \text{if } i < j, \\ X_j & \text{if } i > j. \end{cases}$$

Then

$$V \in {}^{\mathbf{k}\mathbb{Z}_m^\theta}_{\mathbf{k}\mathbb{Z}_m^\theta}\mathcal{YD}^{\Phi_{\underline{a}}}$$

	Arithmetic root systems	m
1.	Rank 2: cases 1–4, 10; rank 3: cases 1–8, 10; rank 4: cases 1–14, rank $\geq$ 5: cases 1–4, 7–10, 22	$m = \Upsilon(|q|)$ if $2 \mid |q|$; $m = 2\Upsilon(|q|)$ if $2 \nmid |q|$
2.	Rank 2: case 5	$m = \Upsilon(|q|)$ if $3 \mid |q|$; $m = 3\Upsilon(|q|)$ if $3 \nmid |q|$
3.	Rank 2: cases 7–9	$m = 6$
4.	Rank 2: case 11	$m = 4$
5.	Rank 2: case 12	$m = 12$
6.	Rank 2: case 14	$m = 10$
7.	Rank 3: case 18	$m = 3$
8.	Rank 4: case 22; rank 5: cases 14, 19	$m = 2$

Table 1.   The number m associated to each arithmetic root system.

where $\underline{a}$ is determined by equations (6.2). According to Proposition 6.3, $\Phi_{\underline{a}}$ is nontrivial and $\mathbb{M} = \mathcal{B}(V)\#\mathbf{k}\mathbb{Z}_m^\theta$ is a genuine pointed Majid algebra such that the arithmetic root system of $\mathcal{B}(V)$ is $\triangle_{\chi,E}$.

Remark 6.5.   The preceding construction provides many new examples of finite-dimensional pointed Majid algebras. It is obvious that $\mathbb{M} = \mathcal{B}(V)\#\mathbf{k}\mathbb{Z}_m^\theta$ is finite-dimensional if and only if $\mathcal{B}(V)$ is finite-dimensional, which is completely determined by its arithmetic root system $(= \triangle_{\chi,E})$. When $\triangle_{\chi,E}$ is of rank 2, or of Cartan type, the associated pointed Majid algebra $\mathbb{M} = \mathcal{B}(V)\#\mathbf{k}\mathbb{Z}_m^\theta$ is finite-dimensional. For other cases, it is an open question whether the corresponding Nichols algebras is finite-dimensional or not.

6.2. Finite-dimensional pointed Majid algebras of Cartan type.   In this subsection, we will give more examples of genuine finite-dimensional pointed Majid algebras. Let $\triangle_{\chi,E}$ be an arithmetic root system. If $\triangle$ is a root system of a complex semisimple Lie algebra, then we call $\triangle_{\chi,E}$ an arithmetic root system of Cartan type.

Definition 6.6.   Let $\mathbb{M}$ be a finite-dimensional connected graded pointed Majid algebra. By Theorem 5.7, there exists a root datum $\mathfrak{D} = \mathfrak{D}(\mathcal{D}_{\chi,E}, \mathbf{S}, \mathbf{X})$ such that $\mathbb{M} = \mathbb{M}(\mathfrak{D})$. If the arithmetic root system of $\mathfrak{D}$ is of Cartan type, then we say that $\mathbb{M}$ is a pointed Majid algebra of Cartan type.

According to [20], if $\mathcal{B}(V)$ is a usual Nichols algebra of diagonal type with finite root system, and the rank of $\mathcal{B}(V)$ is θ, then there exist a bicharacter χ on $\mathbb{Z}^\theta$ and a basis E such that $\triangle(\mathcal{B}(V))_{\chi,E}$ is an arithmetical root system. In fact, arithmetic root systems include more information than root systems of Nichols algebras. For instance, all the rank-1 Nichols algebras have the same root system, i.e., $\{\alpha, -\alpha\}$, but there are both finite-dimensional and infinite-dimensional rank-1 Nichols algebras, hence they have different arithmetic root systems.

Suppose $C = (c_{ij})_{1 \leq i,j \leq \theta}$ is a finite Cartan matrix. We say $1 \leq i \neq j \leq \theta$ are connected if there exist $k_1, k_2, \ldots, k_n$ such that $c_{i,k_1}, c_{k_1 k_2}, \ldots, c_{k_{n-1}k_n}, c_{k_n,j}$ are nonzero numbers. In this subsection, we will prove the following conclusion.

Proposition 6.7. *Suppose that $\triangle$ is a finite root system of Cartan type. Then there exist an abelian group $\mathbb{G}$ and a root datum $\mathfrak{D} = \mathfrak{D}(\mathcal{D}_{\chi,E}, \mathbf{S}, \mathbf{X})$ over $\mathbb{G}$, such that $\mathbb{M}(\mathfrak{D})$ is a finite-dimensional genuine pointed Majid algebra and the root system of $\mathfrak{D}$ is $\triangle$.*

Proof. Let $C = (c_{ij})_{1 \leq i,j \leq \theta}$ be the finite Cartan matrix corresponding to $\triangle$, and let $D = \delta(d_1, \ldots, d_\theta)$ be the diagonal matrix such that DC is symmetric. Let $\mathfrak{J}$ be the set of connected components of $\{1, \ldots, \theta\}$. Fix an order $<$ on $\mathfrak{J}$. For each $I \in \mathfrak{J}$, define a positive odd integer $\mathrm{m}_I > 2$ satisfying the following:

(T1) If $I, I' \in \mathfrak{J}$ and $I < I'$, then $\mathrm{m}_I \,|\, \mathrm{m}_{I'}$.

(T2) If I is of type G_3, we assume that $3 \nmid \mathrm{m}_I$.

Let $q_I = \zeta_{\mathrm{m}_I^2}$ be a primitive m_I^2-th root of unity, and $q_{ii} = q_I^{d_i}$, $i \in I$, $I \in \mathfrak{J}$. Define

$$q_{ij} = q_{ii}^{-d_i c_{ij}}, \quad q_{ji} = 1$$

for all $1 \leq i < j \leq \theta$. Let $E = \{e_1, \ldots, e_\theta\}$ be the canonical basis of $\mathbb{Z}^\theta$ and χ a bicharacter on $\mathbb{Z}^\theta$ given by

$$\chi(e_i, e_j) = q_{ij}, \quad 1 \leq i, j \leq \theta.$$

Then it is obvious that $\triangle_{\chi,E}$ is an arithmetic root system. Set

$$\mathbb{G} = \mathbb{Z}_{\mathrm{m}_1} \times \cdots \times \mathbb{Z}_{\mathrm{m}_\theta} = \langle \mathrm{g}_1 \rangle \times \cdots \times \langle \mathrm{g}_\theta \rangle,$$

where $\mathrm{m}_i = \mathrm{m}_I$, $i \in I$. Next we will show that there exists a root datum $\mathfrak{D} = \mathfrak{D}(\mathcal{D}_{\chi,E}, \mathbf{S}, \mathbf{X})$ over $\mathbb{G}$.

Let $\mathbf{S} = (s_{ij})_{1 \leq i,j \leq \theta}$ be the identity matrix, i.e., $s_{ij} = \delta_{ij}$, $1 \leq i, j \leq \theta$. Then the inverse matrix $T = (t_{ij})$ is also the identity matrix. For all $1 \leq i, j \leq \theta$, define $\mathbf{X} = (x_{ij})_{1 \leq i,j \leq \theta}$ through

$$x_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -d_i c_{ij} & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

Then we have

$$\sum_{j=1}^{m} x_{ij} t_{kj} = x_{ik} \equiv 0 \pmod{\mathrm{m}_i}, \quad 1 \leq k < i \leq n,$$

which implies (4.5).

According to the definition of x_{ij}, it is obvious that

$$\prod_{k=1}^{n} \zeta_{\mathrm{m}_k}^{s_{ik} x_{ki}} = \zeta_{\mathrm{m}_i}^{x_{ii}} = q_{ii},$$

where $m_i = \mathrm{m}_i^2$ for $1 \le i \le \theta$. When $i < j$, we have

$$\prod_{k=1}^{n} \zeta_{m_k}^{s_{ik}x_{kj}+s_{jk}x_{ki}} = q_{ij}^{-x_{ij}} = q_{ij}q_{ji} = \widetilde{q_{ij}}.$$

This implies that $\mathfrak{D} = \mathfrak{D}(\mathcal{D}_{\chi,E}, \mathbf{S}, \mathbf{X})$ is a root datum over $\mathbb{G}$. Since $a_i \equiv x_{ii} = d_i \pmod{\mathrm{m}_i}$, we have $a_i \ne 0$ for each i. Hence $\underline{a}$ is nontrivial, which implies that $\mathbb{M}(\mathfrak{D})$ is genuine.

Finally, we prove that $\mathbb{M}(\mathfrak{D})$ is finite-dimensional. We need to show that for any $\alpha \in \bigtriangleup^+$, the nilpotent index N_α is finite. Let $\bigtriangleup_I$ be the root system corresponding to $I \in \mathfrak{J}$. It is obvious that $\bigtriangleup = \bigcup_{I \in \mathfrak{J}} \bigtriangleup_I$. Let G be the bigger group defined by $\mathbb{G}$ and π, ι; see the sentences before Lemma 4.6. We know that there exists a 2-cocycle J on G such that

$$U = \widetilde{V(\mathfrak{D})}^{J^{-1}} \in {}_{\mathbf{k}G}^{\mathbf{k}G}\mathcal{YD}$$

and $\mathcal{B}(U)$ and $\mathcal{B}(\mathfrak{D})$ have the same root system. Because the nilpotent index of a root vector is invariant under twisted deformation by J, we only need to prove that the nilpotent index of the arithmetic root vector of $\mathcal{B}(U)$ is finite. According to [4, Theorem 5.1] and (T2), for all $\alpha \in \bigtriangleup_I^+$, $I \in \mathfrak{J}$, we have $N_\alpha = N_I = |q_{ii}|$, $i \in I$, hence the nilpotent index must be finite.

This completes the proof of the proposition. □

Remark 6.8. From the proof of Proposition 6.7, we see that there are many choices of $\mathbb{G}$, hence many pointed Majid algebra associated to each arithmetic root system of Cartan type. This also provides a large class of examples of new genuine finite-dimensional pointed Majid algebras.

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